LORENZ-LIKE ATTRACTORS

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Contents

1. Introduction

In this note we will consider a compact finite dimensional boundaryless manifold M of dimensions 1 to 3 and study the dynamics of the flow associated to a given smooth vector field X on M from the topological and measure-theoretic or ergodic point-of-view.

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We fix on M some Riemannian metric which induces a distance dist on M and naturally defines an associated Riemannian volume form Leb which we call *Lebesgue measure* or simply *volume*, and always take Leb to be normalized: $\mathrm{Leb}(M) = 1$.

We always assume that a C^r vector field X on M is given, $r \geq 1$, and consider the associated global *flow* $(X^t)_{t\in\mathbb{R}}$ (since X is defined on the whole of M, which is compact, X is bounded and X^t is defined for every $t \in \mathbb{R}$.) Recall that the flow $(X^t)_{t\in\mathbb{R}}$ is a family of C^r diffeomorphisms satisfying the following properties:

(1) $X^0 = Id : M \to M$ is the identity map of M;
(2) $X^{t+s} = X^t \circ X^s$ for all $t, s \in \mathbb{R}$,

and it is *generated by the vector field* X if

(3) $\frac{d}{dt}X^t(q)\big|_{t=t_0} = X\big(X_{t_0}(q)\big)$ for all $q \in M$ and $t_0 \in \mathbb{R}$.

Note that reciprocally a given flow $(X^t)_{t\in\mathbb{R}}$ determines a unique vector field X whose associated flow is precisely $(X^t)_{t\in\mathbb{R}}$.

In what follows we denote by $\mathfrak{X}^r(M)$ the vector space of all C^r vector fields on M endowed with the C^r topology and by $\mathcal{F}^r(M)$ the space of all flows on M also with the C^r topology. Many times we usually denote

Given $X \in \mathfrak{X}^r(M)$ and $q \in M$, an orbit segment $\{X^t(q); a \leq t \leq b\}$ is denoted by $X^{[a,b]}(q)$. We denote by DX^t *the derivative of* X^t *with respect to the ambient variable* q *and when convenient we set* $D_qX^t = DX^t(q)$. Analogously, DX *is the derivative of the vector field* X with respect to the ambient variable q, and when convenient we write D_qX for the derivative DX at q, $DY(q)$.

An *equilibrium* or *singularity* for X is a point $\sigma \in M$ such that $X^t(\sigma) = \sigma$ for all $t \in \mathbb{R}$, i.e. a fixed point of all the flow maps, which corresponds to a zero of the associated vector field X: $X(\sigma) = 0$. We denote by $S(X)$ the set of singularities (zeroes) of the vector field X. Every point $p \in M \setminus S(X)$, that is p satisfies $X(p) \neq 0$, is a *regular* point for X.

An *orbit* of X is a set $\mathcal{O}(q) = \mathcal{O}_X(q) = \{X^t(q) : t \in \mathbb{R}\}$ for some $q \in M$. Hence $\sigma \in M$ is a singularity of X if, and only if, $\mathcal{O}_X(\sigma) = \{\sigma\}$. A *periodic orbit* of X is an orbit $\mathcal{O} = \mathcal{O}_X(p)$ such that $X^T(p) = p$ for some minimal $T > 0$ (equivalently $\mathcal{O}_X(p)$ is compact and $\mathcal{O}_X(p) \neq \{p\}$). We denote by $\mathrm{Per}(X)$ the set of all periodic orbits of X.

A *critical element* of a given vector field X is either a singularity or a periodic orbit. The set $C(X) = S(X) \cup \mathrm{Per}(X)$ is the set of *critical elements* of X.

We say that $p \in M$ is *non-wandering* for X if for every $T > 0$ and every neighborhood U of p there is $t > T$ such that $X^t(U) \cap U \neq \emptyset$. The set of non-wandering points of X is denoted by $\Omega(X)$. If $q \in M$, we define $\omega_X(q)$ as the set of accumulation points of the positive orbit $\{X^t(q) : t \geq 0\}$ of q. We also define $\alpha_X(q) = \omega_{-X}$, where $-X$ is the time reversed vector field X, corresponding to the set of accumulation points of the negative orbit of q. It is immediate that $\omega_X(q) \cup \alpha_X(q) \subset \Omega(X)$ for every $q \in M$.

A subset Λ of M is *invariant* for X (or X-invariant) if $X^t(\Lambda) = \Lambda$, $\forall t \in \mathbb{R}$. We note that $\omega_X(q)$, $\alpha_X(q)$ and $\Omega(X)$ are X-invariant. For every compact invariant set Λ of X we define the *stable set* of Λ

$$W^s_X(\Lambda) = \{q \in M : \omega_X(q) \subset \Lambda\},$$

and also its *unstable set*

$$W^u_X(\Lambda) = \{q \in M : \alpha_X(q) \subset \Lambda\}.$$

A compact invariant set Λ is *transitive* if $\Lambda = \omega_X(q)$ for some $q \in \Lambda$, and *attracting* if $\Lambda = \cap_{t \geq 0} X^t(U)$ for some neighborhood U of Λ satisfying $X^t(U) \subset U$, $\forall t > 0$. An *attractor* of X is a transitive attracting set of X and a *repeller* is an attractor for $-X$. We say that Λ is a *proper* attractor or repeller if $\emptyset \neq \Lambda \neq M$.

A *sink* of X is a singularity of X which is also an attractor of X, it is a trivial attractor of X. A *source* of X is a trivial repeller of X, i.e. a singularity which is a attractor for $-X$.

A *singularity* σ *is hyperbolic* if the eigenvalues of $DX(\sigma)$, the derivative of the vector field at σ, have a real part different from zero. In particular sinks and sources are hyperbolic singularities, where all the eigenvalues of the former have negative real part and those of the latter have positive real part.

A *periodic orbit* $\mathcal{O}_X(p)$ *of* X *is hyperbolic* if the eigenvalues of $DX^T(p) : T_pM \to T_pM$, the derivative of the diffeomorphism X^T, where $T > 0$ is the period of

We say that a vector field $X \in \mathfrak{X}^r(M), r \geq 1$ is C^s-*structurally stable*, $s \leq r$, if there exists a neighborhood $\mathcal{V}$ of X in $\mathfrak{X}^s(M)$ such that every $Y \in \mathcal{V}$ is topologically equivalent to X.

Roughly speaking, a vector field is structurally stable if its qualitative features are robust under small perturbations.

2. Three dimensional chaotic attractors

In 1963 the meteorologist Edward Lorenz published in the Journal of Atmospheric Sciences [16] an example of a parametrized polynomial system of differential equations

$$\begin{aligned} \dot{x} &= a(y - x) & \qquad a &= 10 \\ \dot{y} &= rx - y - xz & \qquad r &= 28 \\ \dot{z} &= xy - bz & \qquad b &= 8/3 \end{aligned} \tag{1}$$

as a very simplified model for thermal fluid convection, motivated by an attempt to understand the foundations of weather forecast. Later Lorenz [17] together with other experimental researches showed that the equations of motions of a certain laboratory water wheel are given by (1). Hence equations (1) can be deduced directly in order to model a physical phenomenon instead of as an approximation to a partial differential equation.

Numerical simulations for an open neighborhood of the chosen parameters suggested that almost all points in phase space tend to a stranger attractor, called the *Lorenz attractor*. However Lorenz's equations proved to be very

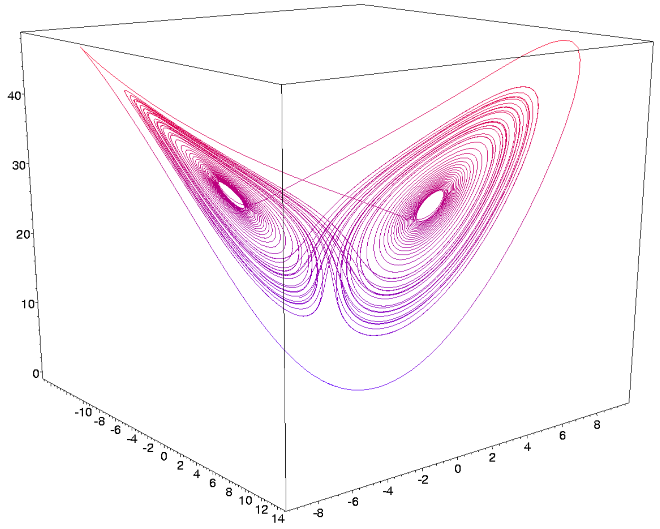

FIGURE 1. Lorenz strange attractor

resistant to rigorous mathematical analysis, and also presented very serious difficulties to rigorous numerical study.

A very successful approach was taken by Afraimovich, Bykov and Shil'nikov [1], and Guckenheimer, Williams [7], independently: they constructed the so-called *geometric Lorenz models* for the behavior observed by Lorenz. These models are flows in 3-dimensions for which one can rigorously prove the existence of an attractor that contains an equilibrium point of the flow, together with regular solutions. The accumulation of regular orbits near a singularity prevents such sets to be hyperbolic. Moreover, for almost every pair of nearby initial conditions, the corresponding solutions move away from each other exponentially fast as they converge to the attractor, that is, the attractor is *sensitive to initial conditions*: this unpredictability is a characteristic of *chaos*. Most remarkably, this attractor is robust: it can not be destroyed by any small perturbation of the original flow.

Another approach was through rigorous numerical analysis. In this way, it could be proved, by [8, 9, 19, 20], that the equations (1) exhibit a suspended Smale Horseshoe. In particular, they have infinitely many closed solutions, that is, the attractor contains infinitely many periodic orbits. However, proving the existence of a strange attractor as in the geometric models is an even harder task, because one cannot avoid the main numerical difficulty posed by Lorenz's equations, which arises from the very presence of an equilibrium point: solutions slow down as they pass near the origin, which means unbounded return times and, thus, unbounded integration errors. In the year 2000 this was finally settled by Warwick Tucker who gave a mathematical proof of the existence of the Lorenz attractor, see [31, 32, 33].

The robustness of this example provides an open set of flows which are not Morse-Smale, nor hyperbolic, and also non-structurally stable.

3. Hyperbolic flows

Let $X \in \mathfrak{X}^r(M)$ be a flow on a compact manifold M. Denote by $m(T) = \inf_{\|v\|=1} \|T(v)\|$ the *minimum norm* of a linear operator T. A compact invariant set $\Lambda \subset M$ of X is *hyperbolic* if

1. admits a continuous DX-invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^X_\Lambda \oplus E^u_\Lambda$, that is we can write the tangent space TxM as a direct sum $E^s_x \oplus E^X_x \oplus E^u_x$, where E^X_x is the subspace in T_xM generated by $X(x)$, satisfying
   - $DX^t(x) \cdot E^i_x = E^i_{X^t(x)}$ for all $t \in \mathbb{R}$, $x \in \Lambda$ and $i = s, X, u$;
2. there are constants $\lambda, K > 0$ such that
   - E^s_Λ *is* (K,λ)*-contracting*, i.e. for all $x \in \Lambda$ and every $t \geq 0$
     $$\|DX^t(x) \mid E^s_x\| \leq K^{-1} e^{-\lambda t},$$
   - E^u_Λ *is* (K,λ)*-expanding*, i.e. for all $x \in \Lambda$ and every $t \geq 0$
     $$m(DX^t \mid E^u) \geq K e^{\lambda t},$$

By the Invariant Manifold Theory [12] it follows that for every $p \in \Lambda$ the sets

$$W^{ss}_X(p) = \{q \in M : \operatorname{dist}(X_t(q), X_t(p)) \xrightarrow[t\to\infty]{} 0\}$$

and

$$W^{uu}_X(p) = \{q \in M : \operatorname{dist}(X_t(q), X_t(p)) \xrightarrow[t\to-\infty]{} 0\}$$

are invariant C^r-manifolds tangent to E^s_p and E^u_p respectively at p. Here dist is the *distance on M induced by some Riemannian norm.*

If $\mathcal{O} = \mathcal{O}_X(p) \subset \Lambda$ is an orbit of X one has that

$$W^s_X(\mathcal{O}) = \cup_{t\in\mathbb{R}} W^{ss}_X(X^t(p)) \quad \text{and} \quad W^u_X(\mathcal{O}) = \cup_{t\in\mathbb{R}} W^{uu}_X(X^t(p))$$

are invariant C^r-manifolds tangent to $E^s_p \oplus E^X_p$ and $E^X_p \oplus E^u_p$ at p, respectively. We shall denote $W^s_X(p) = W^s_X(\mathcal{O}_X(p))$ and $W^u_X(p) = W^u_X(\mathcal{O}_X(p))$ for the sake of simplicity.

A *singularity* (respectively *periodic orbit*) of X is *hyperbolic* if its orbit is a hyperbolic set of X. Note that $W^{ss}_X(\sigma) = W^s_X(\sigma)$ and $W^{uu}_X(\sigma) = W^u_X(\sigma)$ for every hyperbolic singularity σ of X. A sink and a source are both hyperbolic singularities. A *hyperbolic* singularity which is *neither* a sink *nor* a source is called a *saddle.*

A hyperbolic set Λ of X is called *basic* if it is transitive and *isolated*, that is $\Lambda = \cap_{t\in\mathbb{R}} \overline{X^t(U)}$ for some neighborhood U of H. It follows from the Shadowing Lemma [23] that every hyperbolic basic set of X either reduces

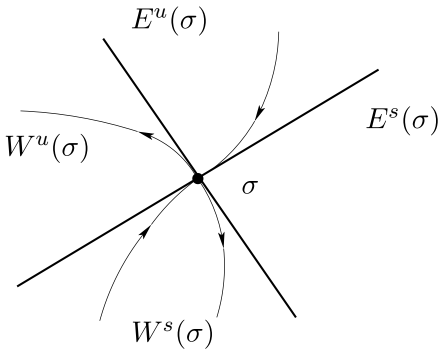

FIGURE 2. A saddle singularity σ for bi-dimensional flow.

to a singularity or else has no singularities and it is the closure of its periodic orbits.

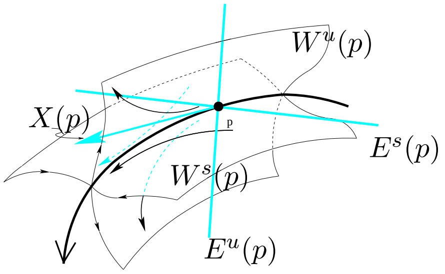

FIGURE 3. The flow near a hyperbolic saddle periodic orbit through p.

We say that X is *Axiom A* if the non-wandering set $\Omega(X)$ is both hyperbolic and the closure of its periodic orbits and singularities. The *Spectral Decomposition Theorem* asserts that if X is Axiom A, then there is a disjoint decomposition $\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_k$, where each Λ_i is a hyperbolic basic set of X, $i = 1, \cdots, k$.

A *cycle* of a Axiom A vector field X is a sub-collection $\{\Lambda_{i_0}, \cdots, \Lambda_{i_k}\}$ of $\{\Lambda_1, \cdots, \Lambda_n\}$ such that $i_0 = i_k$ and $W^u_X(\Lambda_{i_j}) \cap W^s_X(\Lambda_{i_{j+1}}) \neq \emptyset$, $\forall 0 \leq j \leq k-1$.

Hyperbolic sets and singularities. The continuity of the DX-invariant splitting on the tangent space of a uniformly hyperbolic set Λ is a consequence of the uniform expansion and contraction estimates (see e.g. [25]). This means that if $x_n \in \Lambda$ is a sequence of points converging to $x \in \Lambda$, and we consider orthonormal basis $\{e^n_i\}_{i=1,\ldots,\dim E^s(x_n)}$ of $E^s(x_n)$, $\{f^n_i\}_{i=1,\ldots,\dim E^u(x_n)}$ of $E^u(x_n)$ and $X(x_n)$ of $E^X(x_n)$, then these vectors converge to a basis of $E^s(x), E^u(x)$ and $E^X(x)$ respectively. In particular the dimension of the subspaces in the hyperbolic splitting is constant if Λ is transitive.

This shows that a uniformly hyperbolic basic set Λ cannot contain singularities, except if Λ is itself a singularity. Indeed, if $\sigma \in \Lambda$ is a singularity then it is hyperbolic but the dimension of the central sub-bundle is zero since the flow is zero at σ. Therefore the dimensions of either the stable or the unstable direction at σ and those of a transitive regular orbit in Λ do not match.

In other words *an invariant subset* Λ *containing a singularity accumulated by regular orbits cannot be uniformly hyperbolic.*

4. Singular cycles and robust singular attractors

A cycle Γ for a flow X^t is a finite sequence $\{\sigma_i, 0 \leq i \leq n\} \subset C(X)$ of hyperbolic critical elements of X^t, with $\sigma_0 = \sigma_n$, such that $W^u(\sigma_j) \cap W^s(\sigma_{j+1}) \neq \emptyset$ for $0 \leq j \leq n$, that is the unstable manifold of one element intersects the stable manifold of the next element. A cycle is *singular* if at least one of its critical elements is a fixed point of X^t.

Cycles play an important role in the bifurcating theory of Dynamical Systems. A singular cycle is one of the mechanisms to go from a Morse-Smale flow (whose non-wandering set is a finite collection of hyperbolic critical elements) to a hyperbolic flow (whose non-wandering set is a finite collection of basic sets) through a one parameter family of flows.

In this chapter we shall describe two types of singular cycles, that will be used in the sequel. Nowadays the first one, presented in Section 5, is denominated *singular-horseshoe*. It was introduced by Labarca and Pacifico in [13] as a model for stable non hyperbolic flows in the context of boundary manifolds. We show that this set satisfies some properties which, will be defined as singular-hyperbolicity.

Finally the third one is the Lorenz geometrical model, introduced by Guckenheimer and Williams [7] and presented in Section 6. This is a model for a robust attractor with singularities for a 3-flow.

5. Singular horseshoe

We start with the description of a map defined on a rectangle into itself which resembles the Smale horseshoe map [28]. For this reason this type of map is nowadays denominated singular horseshoe.

Afterward, we exhibit a singular cycle presenting a singular horseshoe map as a first return map. Then we show in several stages that the singular horseshoe is a transitive partially hyperbolic set with volume expanding central direction.

5.1. **A singular horseshoe map.**

Given $\delta > 0$ small enough, $\lambda < 1/2$ and $\mu > 1$, let $Q = [-1, 0] \times [0, 1+\delta]$ and define

$$R_\delta = Q \setminus \left((\mu^{-1}(1+\delta), 1/2 - \delta) \times (1/2, 1) \right).$$

Let $F : R_\delta \to Q$, $(x, y) \mapsto (g(x, y), f(y))$ be a smooth map satisfying:

(a) $|\partial_x g(x,y)| < 1/2$ for all $(x,y) \in R_\delta$ and

$$g(x,y) = \lambda \cdot x \quad \text{for} \quad 0 \le y \le \mu^{-1}(1+2\delta)\,.$$

(b) $f : I \setminus (J \cup K) \to I$ where $I = [0,1]$, $J = (\mu^{-1}(1+2\delta), 1/2-\delta)$ and $K = (1/2, 1)$ satisfying
  (i) $f(y) = \mu \cdot y$ for $0 \le y \le \mu^{-1}(1+2\delta)$,
  (ii) $f'(y) >> \mu$ for $y \in [1/2-\delta, 1/2] \cup [1, 1+\delta]$.

(c) $F(x,1) = F(x,1/2) = (\alpha, 0)$ for $-1 \le x \le 0$ with a fixed $-1 < \alpha < \lambda$.

(d) the following sets

$$\gamma_{-1} = F(\{-1\} \times (1, 1+\delta]), \qquad \gamma_0 = F(\{0\} \times (1, 1+\delta]),$$
$$\beta_0 = F(\{0\} \times [1/2-\delta, 1/2)), \qquad \beta_{-1} = F(\{-1\} \times [1/2-\delta, 1/2))$$

are disjoint C^1 curves, except for the point $(\alpha, 0)$ where all are tangent. These curves are contained in $(-1, -\lambda) \times [0, 1+\delta]$ and are transverse to the horizontal lines. Moreover, if $d(A,B)$ denotes the distance between the sets A and B, and $L = \{-1\} \times [0, 1+\delta]$ then

$$d(\gamma_{-1}, L) < d(\gamma_0, L) < d(\beta_0, L) < d(\beta_{-1}, L).$$

Figure 4 displays the main features of the map F.

Observe that, by construction, the horizontal lines $\{x\} \times [0, 1+\delta]$, for $x \in [-1, 0]$, are invariants by F. They are also uniformly contracted by a factor $0 < c_0 < 1/2$. This guaranties that Q has a uniformly contracted (strong-)stable foliation invariant by F that we denote by $\mathcal{F}^{ss}(Q)$.

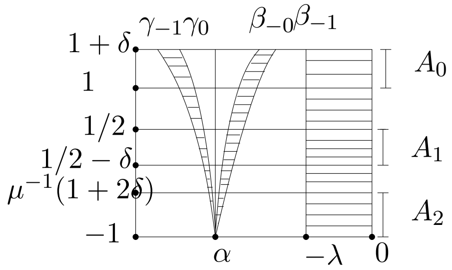

FIGURE 4. A singular-horseshoe map

Define the following rectangles

$$A_0 = [-1, 0] \times [1, 1+\delta], \qquad A_1 = [-1, 0] \times [1/2-\delta, 1/2],$$
$$A_2 = [-1, 0] \times [0, \mu^{-1}(1+2\delta)].$$

Note that

$$R_\delta = \bigcup_{i=0}^{i=1} A_i \quad \text{and define} \quad \Omega_F = \bigcap_{n \in \mathbb{Z}} F^n(R_\delta)\,.$$

It is clear that $F^{-1}(\Omega_F) = \Omega_F$.

5.1.1. *Singular symbolic dynamics.* We now associate a symbolic dynamics to the restriction $F \mid \Omega_F$. For this, consider a map $\widetilde{F} : R_\delta \to Q$ such that $\widetilde{F}$ has the same properties described for F, except that $\widetilde{F}([-1,1] \times \{1\})$ and $\widetilde{F}([-1,1] \times \{1/2\})$ are disjoint intervals I and J contained in the interior of $[-1,\lambda] \times \{0\}$ as in Figure 5. Define $\Omega = \cap_{n\in\mathbb{Z}}\widetilde{F}^n(R_\delta)$.

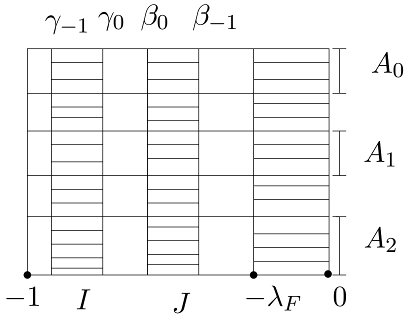

Figure 5. A Smale horseshoe map

Clearly $\widetilde{F}$ is a Smale horseshoe map. Roughly speaking, F is obtained from $\widetilde{F}$ pinching the intervals I and J into a unique point in such a way that the resulting boundary lines $\widetilde{\gamma}_{-1}, \widetilde{\gamma}_0, \widetilde{\beta}_0$, and $\widetilde{\beta}_{-1}$ are tangent at this point.

Let Σ^3 be the set of doubly infinite sequences of symbols in $\{0,1,2\}$ endowed with the topology given by the distance

$$d(x,y) = \sum_{i\in\mathbb{Z}} \frac{|x_i - y_i|}{3^{|i|}}$$

and $\sigma : \Sigma^3 \to \Sigma^3$ be the left shift map $\sigma(x)_i = x_{i+1}$.

It is well known (see e.g. [28] but also the textbooks of e.g. Devaney [4] or Robinson [26]) that there exists a homeomorphism $\widetilde{H} : \Omega \to \Sigma^3$ which conjugates $\widetilde{F}$ and σ, i.e. $\widetilde{H} \circ \widetilde{F} = \sigma \circ \widetilde{H}$. The image $\widetilde{H}(x)$ of $x \in \Omega$ is the sequence $(\widetilde{H}(x)_i) \in \Sigma^3$ defined by

$$\widetilde{H}(x)_i = j \in \{0,1,2\} \iff \widetilde{F}^i(x) \in A_j, \quad i \in \mathbb{Z}. \tag{2}$$

Recall that the set of periodic orbits for σ is dense in Σ^3 and that there exists a dense orbit.

We now describe the sequences associated, in a similar way, to points Ω_F.

Observe that the tangency point $(\alpha, 0)$ is the unique point of Ω_F outside of $[-\lambda, 0] \times [0, 1+\delta]$ which remains forever in the bottom boundary of Q. This line corresponds to the local stable manifold of the fixed point $(0,0)$ of F.

- Since $[-1,0] \times \{0\} = \cap_{n\le 0}\widetilde{F}^n(A_2)$ we have $z \in [-1,0] \times \{0\} \cap \Omega$ if, and only if, $\theta_i(z) = 2$ for all $i \ge 0$, i.e. $\widetilde{H}(z) = (\dots, x_{-1}, 2, 2, 2, \dots)$.

The points belonging to this line which are outside of $[-\lambda, 0] \times [0, 1+\delta]$ are the points of the local stable manifold of $(0,0)$ which are different from $(0,0)$, i.e. their corresponding codes differ from the constant sequence $x_i \equiv 2$ at some coordinate with negative index. Defining Σ^3_* the subset of Σ^3 of those sequences $(x_i)_{i\in\mathbb{Z}}$ with $x_0 \in \{0,1\}$ and $x_i = 2$ for all $i \geq 1$, then

$$W^s_{\widetilde{F}}\big(\widetilde{H}(0,0)\big) \setminus \widetilde{H}\big([-\lambda,0]\times[0,1+\delta]\big) = \bigcup_{k\geq 1} \sigma^k \Sigma^3_* = \widetilde{\Sigma}^3_*$$

Note that $\sigma^{-1}\widetilde{\Sigma}^3_* \subseteq \widetilde{\Sigma}^3_*$. Defining an equivalence relation on Σ^3 by $\theta \sim \widetilde{\theta}$ if and only if $\theta, \widetilde{\theta} \in \widetilde{\Sigma}^3_*$, then this relation is preserved by the shift.

Let $\widetilde{\Sigma}^3$ be the corresponding quotient space and $\widetilde{\sigma}$ the associated quotient shift map. This map can be seen as the original full shift map on three symbols after identifying the sequences on $\widetilde{\Sigma}^3_*$, which correspond to the points which are taken to $(\alpha, 0)$ by F.

By the above considerations and the dynamics of F we get

Lemma 5.1. *There exists a homeomorphism $H_F : \Omega_F \to \widetilde{\Sigma}^3$ which conjugates $F \mid \Omega_F$ and H_F, that is $H_F \circ (F \mid \Omega_F) = \widetilde{\sigma} \circ H_F$.*

The homeomorphism H_F is defined just as in (2) replacing $\widetilde{F}$ by F.

Observe that the set of periodic orbits for $\widetilde{\sigma}$ is the same set of periodic orbits for σ. Note also that the dense orbit for σ is not contained in $\widetilde{\Sigma}^3_*$. Therefore the set of periodic orbits for $\widetilde{\sigma}$ is dense in $\widetilde{\Sigma}^3$ and this space contains a dense orbit. The existence of the conjugation above ensures that Ω_F has a dense subset of periodic orbits and a dense orbit for the dynamics of F.

5.2. **A singular cycle with a singular horseshoe first return map.** We start by giving a definition of a special type of singularity of a vector field X in a 3-manifold.

Definition 1. We say that a singularity σ of a 3-flow X^t is Lorenz-like if the eigenvalues λ_i, $1 \leq i \leq 3$ are real and satisfy

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1. \tag{3}$$

Next we shall exhibit a singular cycle $\mathcal{C}$ having a Lorenz-like singularity p and a hyperbolic saddle-type closed orbit σ, connected through a branch of the unstable manifold associated to p: this branch is contained in the stable manifold associated to σ. Moreover there are two orbits of transverse intersection between $W^s(p)$ and $W^u(\sigma)$. The cycle will be constructed in such away that it is contained in the maximal invariant set $\Lambda(X)$ of a vector field X in a neighborhood U of $\mathcal{C}$, and the first return map associated to $\mathcal{C}$ is a singular horseshoe map, see Figure 6.

We start with a vector field $X_0 \in \mathfrak{X}^r(\mathbb{D}^3)$ on the 3-disk $\mathbb{D}^3$ in $\mathbb{R}^3$. This vector field has one repeller singularity r_1 at the north pole. Outside a neighborhood of r_1, X_0 has four singularities which we denote by p, p_1, p_2, r_2, plus a hyperbolic closed orbit σ. These satisfy the following:

(1) p is a Lorenz-like singularity.

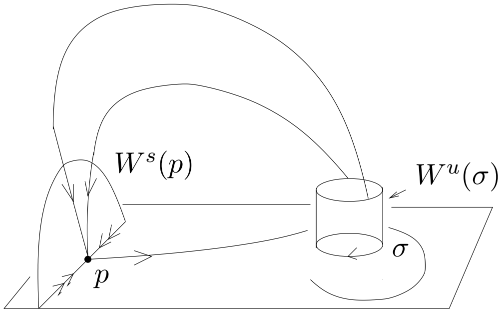

FIGURE 6. A singular cycle

(2) (p,σ) is a saddle connection with a branch $\gamma^u(p)$ of $W^u(p) \setminus \{p\}$ whose ω-limit set is σ. By the Hartman-Grobman Theorem there exists a neighborhood $p \in \mathcal{N} \subset \mathbb{R}$ such that the restriction of X_0 to $\mathcal{N}$ is equivalent to the linear vector field $L(x_1, x_2, x_3) = (\lambda_2 x_1, \lambda_1 x_2, \lambda_3 x_3)$.
(3) p_1 is an attractor and is also the ω-limit set of the other branch of $W^u(p) \setminus \{p\}$.
(4) p_2 is an attractor and is the ω-limit of $W^u(\sigma) \setminus \{\sigma\}$.
(5) r_2 is a repeller contained in the interior of the 2-disk $\mathbb{D}^2$ bounded by σ in $\mathbb{S}^2$.
(6) We assume that
    (a) $p_1, p, \gamma^u(p), \sigma$ and $\mathbb{D}^2$ are contained in the boundary $\partial(\mathbb{D}^3) = \mathbb{S}^2$ of the 3-disk;
    (b) the eigenvalues of $DX_0(r_2)$ corresponding to eigenvectors in $T\mathbb{S}^2$ are complex conjugates. Therefore the part of $W^u(r_2) \setminus \{r_2\}$ in $\mathbb{S}^2$ is a spiral whose ω-limit set is σ.
    (c) the strong unstable manifold $W^{uu}(r_2) \setminus \{r_2\}$ is contained in the interior of $\mathbb{D}^3$ and its ω-limit set is the attractor p_2.
(7) The α-limit set of $W^s(p) \setminus \{p\}$ is the repeller r_1 and $W^s(p)$ separates the two attractors.

Figure 7 shows the essential features of the vector field X_0 outside a neighborhood of r_1. Observe that X_0 constructed in this way is a Morse-Smale vector field.

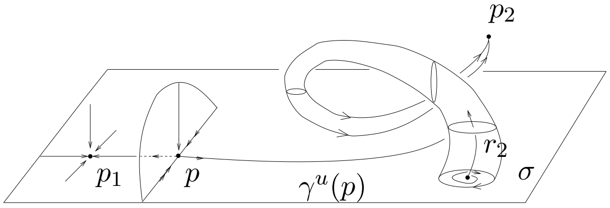

FIGURE 7. The vector field X_0.

Now we can modify the vector field X_0 away from its critical elements, in particular away from the neighborhood $\mathcal{N}$ of p, in order to produce a unique tangency between $W^s(p)$ and $W^u(\sigma)$, see Figure 8.

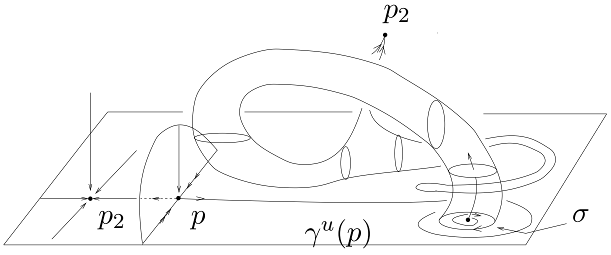

FIGURE 8. Producing a unique tangency.

By another slight perturbation of the above vector field we get a vector field X such that $W^u(\sigma)$ is transverse to $W^s(p)$ at two orbits, see Figure 9.

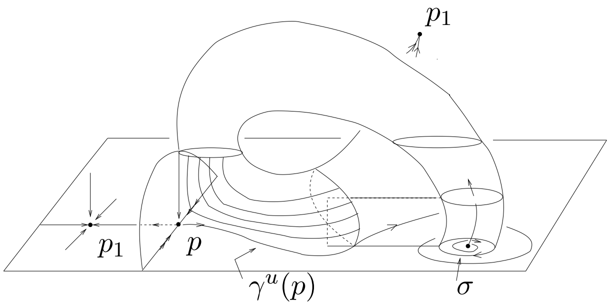

FIGURE 9. One point of tangency

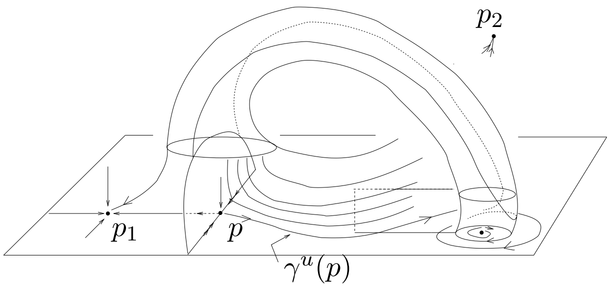

FIGURE 10. The final vector field.

5.2.1. *The first return map associated to $\mathcal{C}$ is a singular horseshoe map.* Now we study the first return map associated to $\mathcal{C}$ and show that it is a singular horseshoe map.

Let S be a cross section to X at $q \in \sigma$. Reparametrizing X, if necessary, we can assume that the period of σ is equal to one and that S is invariant by X^1: there exists a small neighborhood $U \subset S$ of q such that $X^1(S \cap U) \subset S$.

Since there are two orbits of transverse intersection of $W^u(\sigma)$ with $W^s(p)$ and the branch $\gamma^u(p)$ has σ as ω-limit set, there exists a first return map F defined on subsets of S, taking points of S back to S under the action of the flow. The goal now is to describe F.

From now we assume mild non-resonant conditions on the eigenvalues of p to ensure that there are C^1 linearizing coordinates (x_1, x_2, x_3) in a neighborhood U_0 containing p.

Let $D^s(p) \subset U_0$ and $D^u(p) \subset U_0$ be fundamental domains for the action of the flow inside $W^s(p)$ and $W^u(p)$ respectively. That is $D^s(p)$ is a circle in $W^s(p) \setminus \{p\}$ containing p in its interior and transverse to X, and $D^u(p)$ is a pair of points, one in each branch of $W^u(p) \setminus \{p\}$.

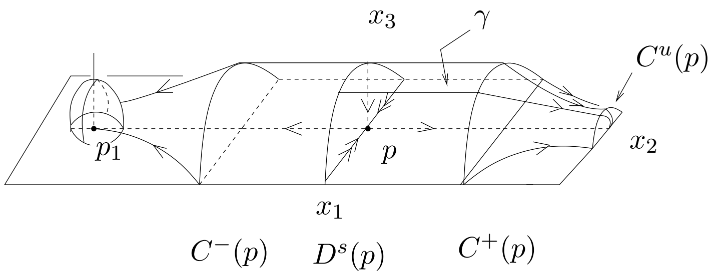

FIGURE 11. The cross section C^s at p.

Let $C^s(p) \subset U_0$ be a cross section to X, as in Figure 11, with several components: $C^s = C^s(p) = C^+(p) \cup D^s(p) \cup C^-(p)$. We assume that $C^-(p)$ is contained in the stable manifold of the attractor p_1. We also assume that the plane $\{x_1 = 0\}$ is a center-unstable manifold for p and we denote it by $W^{cu}(p)$. Let $C^u(p)$ be a cross section to X formed by a 2-disk through the point of $\gamma^{(}p) \cap D^u(p)$.

Observe that if γ is a C^1 curve transverse to $D^s(p)$ and $\gamma \cap W^{ss}(p) = \emptyset$, then

$$C^u(p) \cap \Big(\bigcup_{t \geq 0} X^t(\gamma) \Big)$$

is a C^1 curve tangent to $W^{cu}(p) \cap C^u(p)$ at $D^u(p) \cap \gamma^u(p)$.

Let $D^s(p_2) \subset \mathbb{D}^3$ be a fundamental domain for the dynamics on $W^s(p_2)$, i.e. the boundary of a 3-ball containing p_2. Let $V \subset S$ be a small neighborhood of $q \in \sigma$, where we have C^1 linearizing coordinates (x, y) for the Poincaré first return map F associated to σ. The eigenvalues of $DF(q)$ are λ, μ both bigger than 1.

Let $Q = [-1, 1] \times [0, 1]$ be a rectangle contained in the interior of V. Assume that

$$[-1, 1] \times \{\frac{1}{2}, 1\} \subset W^s(p) \quad \text{and} \quad [-1, 1] \times \{0\} \subset \mathbb{S}^2.$$

There are only two orbits of transverse intersection between $W^u(\sigma)$ and $W^s(p)$, and the points in $\{1\} \times (1/2, 1)$ will fall in the stable set of p_1, by construction of the vector field X. Since $W^s(p_1)$ is open we can assume that $[-1,1]\times(1/2,1)\subset W^s(p_1)$ (taking V small enough) and also

$$X^1([-1,1]\times(1/2,1))\subset C^-(p)$$

through a reparametrization of time if necessary. Assume further that there exists $\delta>0$ such that $(1+2\delta)\mu^{-1}<1/2-\delta$ and

(a) for $A_0=[-1,1]\times(1,1+\delta]$ we have $X^1(A_0)\subset C^+(p)$;
(b) for $A_1=[-1,1]\times[1/2-\delta,1]$ we have $X^1(A_1)\subset C^+(p)$;
(c) $X^2\big([-1,1]\times[1+\delta,1+2\delta]\big)\subset D^s(p_2)$;
(d) $X^2\big([-1,1]\times[1/2-2\delta,1/2-\delta]\big)\subset D^s(p_2)$;
(e) for $A_2=[-1,1]\times[0,(1+2\delta)\mu^{-1}]$ we have

$$X^1(A_2)=[-\lambda,0]\times[0,1+2\delta]\subset Q.$$

Now define

$$H_1(X)=\bigcup_{t\geq 0}X^t(X^1(A_0))\cap C^u(p),\quad H_2(X)=\bigcup_{t\geq 0}X^t(X^1(A_1))\cap C^u(p)\,.$$

Clearly $H_i(X)$ are cones tangent to $W^{cu}(p)\cap C^u(p)$ at $D^u(p)$ for $i=1,2$, see Figures 12 and 13.

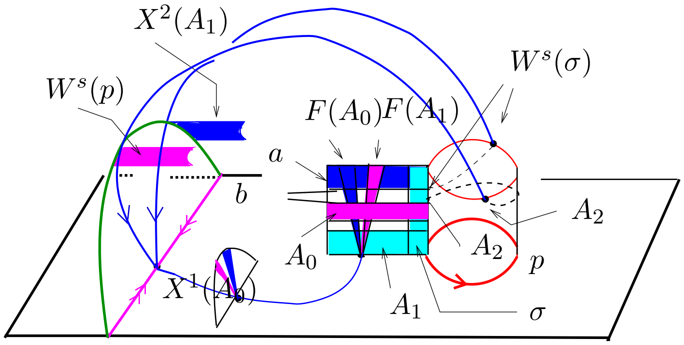

FIGURE 12. The first return map to Q.

Let α be the first intersection point between $W^u(p)$ and Q. We can assume that $\tau_i(X)=X^1(H_i(X))$ is contained in Q and that these sets are cones tangent to $W^{cu}(p)\cap Q$ at α, for $i=1,2$.

Clearly we can also assume that

$$X^3\big([-1,1]\times\{1+\delta\}\big)\subset\tau_1(X)\quad\text{and}\quad X^3\big([-1,1]\times\{\frac{1}{2}-\delta\}\big)\subset\tau_2(X).$$

If necessary, we modify the vector field X in order to have (see Figure 14):

(a) horizontal lines $\{y=\mathit{constant}\}$ going to horizontal lines in $\tau_i(X)$;

(b) writing π_y for the projection on the y-axis in V

$$\pi_y\Big(X^3\big([-1,1]\times\{1+\delta\}\big)\Big)=\{1+2\delta\}\quad\text{and}$$
$$\pi_y\Big(X^3\big([-1,1]\times\{\frac{1}{2}-\delta\}\big)\Big)=\{1+2\delta\};$$

(c) for $D^s_\sigma=[-1,\lambda]\times[0,1+\delta]$ we have $\tau_i(X)\subset \operatorname{int} D^s_\sigma$ for $i=1,2$.

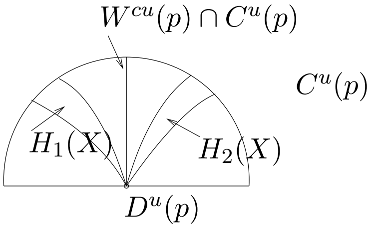

FIGURE 13. The first return map at $D^u(p)$

Now we describe the first return map F.

- If we take a point (x,y) with $1+\delta<y\le 1+2\delta$, then (x,y) is contained in the stable manifold of the attractor p_2 and F is not defined at these points.
- For either a point $(x,1)\in Q$ or $(x,1/2)\in Q$ we define $F(x,1)=\alpha=F(x,1/2)$.
- For points $(x,y)\in Q$ with $0\le y\le \mu^{-1}(1+2\delta)$ we define $F(x,y)=(\lambda x,\mu y)$.
- For points (x,y) such that either $1<y\le 1+\delta$ or $1/2-\delta\le y<1/2$, we define $F(x,y)$ as the first intersection of the positive orbit through (x,y) with the rectangle $Q_\delta=[-1,1]\times[0,1+2\delta]$.
- For points (x,y) with $1/2<y<1$ the first return F is not defined, since these points are in the stable manifold of the attractor p_1.
- F is also not defined for points (x,y) with $\mu^{-1}(1+2\delta)<y<1/2-\delta$. Indeed, these points are such that the projection on the y-axis of their first return to S is larger than $1+2\delta$. So these points return once to S and then they are taken to the attractor p_2.

Then the first return map F has the expression:

$$F(x,y)=\left\{\begin{array}{cll}(\lambda x,\mu y) & \text{if} & 0\le y\le \mu^{-1}(1+2\delta)\\ (g_1(x,y),f_1(y)) & \text{if} & 1\le y\le 1+\delta\\ (g_2(x,y),f_2(y)) & \text{if} & 1/2-\delta\le y\le 1/2\end{array}\right.$$

with

- $g_i(x,y)$ is some smooth function with $\left|\partial_x g_i\right|<c<\frac{1}{2}$, and
- f_i is a smooth function satisfying $f_i'(y)>\mu$ and $0\le f_i(y)\le 1+2\delta$, for $i=1,2$.

We assume that the image $F(\{0\}\times[0,1+\delta])$ is transverse to the horizontal lines in Q_δ.

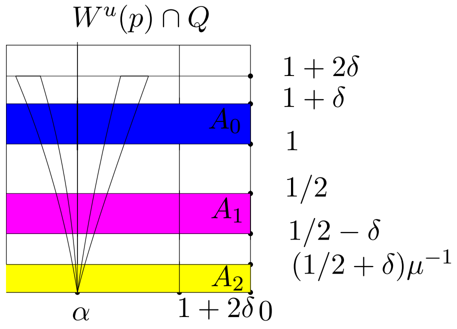

Figure 14. The singular horseshoe return map.

The non-trivial dynamics of F is concentrated in the square Q_δ.

Let $\Omega_F = \cap_{n\geq 0}\overline{F^n(Q_\delta)}$. Observe that the non-wandering set $\Omega(X)$ is the disjoint union of the critical elements $\{r_1, r_2, p_1, p_2\}$ and Λ, where Λ is the closure of the saturation by the flow X^t of the non-wandering set of the first return map F described above, i.e. $\Lambda = \overline{\cup_{t\geq 0} X^t(\Omega_F)}$.

The set Λ is the maximal invariant set containing the singular cycle $\mathcal{C}$ in the neighborhood U chosen at the beginning of the construction. This invariant set is the so called *singular horseshoe*.

Remark 5.2. On the boundary of the manifold $\mathbb{D}^3$, which is preserved by the flow, we have a Morse-Smale system. Hence any vector field Y close to X preserving the boundary will have the same features as X on the boundary.

Moreover the features of X depend on the transverse intersection of certain invariant manifolds of the hyperbolic critical elements, all of which lie on the boundary of the ambient manifold. Hence every vector field close to X preserving the boundary will exhibit the same critical elements and the same transversality relations between them, so *the singular-horseshoe is robust among the vector fields which preserve the boundary of* $\mathbb{D}^3$.

5.3. **The singular horseshoe is a partially hyperbolic set with volume expanding central direction.** We start by constructing local stable and unstable manifolds through points of Ω_F with respect to F. The stable and unstable foliation of the singular horseshoe Λ is the obtained as the saturation by the flow of these manifolds. Then we explain how to obtain the strong-stable foliation. Having these foliations we can define a splitting of the tangent space at Λ which will behave much like a hyperbolic splitting.

5.3.1. *Stable manifold for points in* Ω_F. Let $F : Q_\delta \to Q$ be the singular horseshoe map defined in the previous subsection.

It is easy to see that any horizontal line crossing Q is uniformly contracted by a factor of $c \in (0, 1/2)$ by the definition of F. Then, given any pair of points x, y of Ω_F in the same horizontal line one has

$$\operatorname{dist}\left(F^k(x), F^k(y)\right) \leq c^k \xrightarrow[k\to+\infty]{} 0.$$

Hence these curves are the local stable manifolds through points of Ω_F with respect to F. Saturating these curves by the flow we obtain the foliation of stable manifolds $\mathcal{F}^s$ through the points of the singular horseshoe.

For the particular case of the saddle singularity p and the periodic orbits σ the stable leaves are given by the stable manifolds of these hyperbolic critical points.

5.3.2. *Unstable manifolds for points of* Ω_F. Define $R_0 = Q \cap F(A_0)$, $R_1 = Q \cap F(A_1)$ and $R_2 = Q \cap F(A_2)$. Then R_0 and R_1 are, except for their vertexes, disjoint cones. R_2 is a rectangle, crossing Q from bottom to top.

For each $i, j \in \{0, 1, 2\}$, let $R_{ij} = R_i \cap F(R_j)$. Then $F(R_j) = \cup_{i=0}^{2} R_{ij}$. Since $F(x, y) = (g(x, y), f(y))$ with $|g_x(x, y)| < c < 1/2$, we have that the horizontal lines are contracted by a factor of c when iterated by F. Thus, except for R_{22} (which is a rectangle strictly contained in R_2), R_{ij} is a cone strictly contained in R_i.

Inductively, given any sequence of n-symbols $x_1, x_2 \cdots, x_n$ with $x_i \in \{0, 1, 2\}$ and $n \geq 2$ define $R_{ix_1x_2\cdots x_n} = R_i \cap F(R_{x_1\cdots x_n})$ for $i = 0, 1, 2$. Then

$$F(R_{x_1\cdots x_n}) = \bigcup_{i=0}^{2} R_{ix_1\cdots x_n}\,.$$

Note that

- If all the x_i are equal to 2, then
$$R_2, \quad R_2 \cap F(R_2), \quad \cdots, \quad R_2 \cap F(R_2) \cap \cdots \cap F^n(R_2)$$
is a strictly decreasing sequence of rectangles converging, in the C^1 topology, to the vertical line $\{0\} \times [0, 1 + \delta]$.
- If there is any $x_{i_0} \in \{0, 1\}$, then the sequence
$$R_{x_0}, \quad R_{x_0} \cap F(R_{x_1}), \quad \cdots, \quad R_{x_0} \cap F(R_{x_1}) \cap F^2(R_{x_2}) \cap \cdots \cap F^n(R_{x_n})$$
is a strictly decreasing sequence of C^1-cones. Hence this sequence converges to a C^1 curve, denoted by $\gamma(x_0, x_1, \cdots)$, which crosses Q from top to bottom, that is, γ intersects each horizontal line of Q in a unique point, see Figure 15.

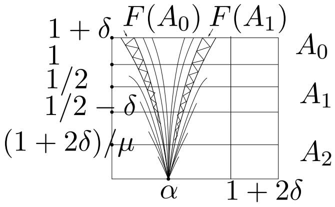

FIGURE 15. The unstable curves of Ω_F tangent at $(\alpha, 0)$.

Note that every point $x \in \Omega_F \setminus \{(\alpha, 0)\}$ has a corresponding code $H_F(x)$ in $\tilde{\Sigma}^3$ whose coordinates with positive index define a unique regular curve $\gamma = \gamma(x_1, x_2, \dots)$ as above. This curve γ is the same for every $y \in \Omega_F$

having a code $H_F(y)$ with the same coordinates as $H_F(x)$ at positive indexes. Such points y form the unstable manifold of x with respect to F, since $d\big(\sigma^{-k}H_F(x), \sigma^{-k}H_F(y)\big) \xrightarrow[k\to+\infty]{} 0$.

Indeed, from the description of the map F, it is clear that γ is expanded by all iterates of F whenever its image is defined. Or, reversing time, by the construction of γ, the pre-image of any pair of points $y, z \in \gamma$ by F^k is well defined for all $k \geq 1$ and, moreover, for any pair y_{-k}, z_{-k} of such pre-images under the same sequence of inverse branches of F satisfies

$$\operatorname{dist}\big(y_{-k}, z_{-k}\big) \leq c^k \xrightarrow[k\to+\infty]{} 0.$$

Saturating these curves by the flow we obtain the central-unstable foliation $\mathcal{F}^u$ through the points of Λ.

The point $(\alpha, 0)$ has already a well defined unstable manifold: the vertical line crossing Q through $(\alpha, 0)$, corresponding to the intersection of the unstable manifold of the orbit of $W^u(p)$ connecting the saddle singularity p to the periodic orbit σ, see Figure 12.

In addition, the saddle singularity p and the periodic orbit σ also have a well defined unstable foliation compatible with the leaves defined above.

5.3.3. *Strong-stable foliation for the singular-horseshoe.* The previous observations show that *every periodic orbit of F on Ω_F is hyperbolic of saddle-type.* Since F is the Poincaré first return map to Q of the flow X, we deduce that *every periodic orbit of X in Λ is hyperbolic of saddle-type.* Moreover the density of periodic orbits for $F \mid \Omega_F$ implies that *the family of periodic orbits of X in Λ is dense in Λ.*

In addition the stable foliation of the periodic orbits coincides with the stable foliation defined above for all points, including the singularity p and the periodic orbit σ. Hence the strong-stable leaves $\widetilde{\mathcal{F}}^{ss}$ defined on the periodic orbits extend (by continuity and coherence) to a strong-stable foliation $\mathcal{F}^{ss}$ defined throughout Λ. Notice that at the singularity p the strong-stable foliation coincides with its strong-stable manifold corresponding to the weakest contracting eigenvalue.

5.3.4. *Partial hyperbolicity.* The flow invariance of the stable $\mathcal{F}^s$, strong-stable $\mathcal{F}^{ss}$ and unstable $\mathcal{F}^u$ foliations through points of Λ and the smoothness of their leaves enables us to define the following DX invariant subbundles: for every point $z \in \Lambda$

$$E_z = T_z\mathcal{F}^{ss}(z) \quad \text{and} \quad F_z = T_z\mathcal{F}^u(z)$$

satisfy $DX^t \cdot E_z = E_{X^t(z)}$ and $DX^t \cdot F_z = F_{X^t(z)}$, for all $t \in \mathbb{R}$.

Now we show that the flow X contracts E uniformly, and contracts strongly than any contraction along F. Then we conclude by showing that X expands volume along F.

Let V be a neighborhood of p where linearizing coordinated are defined. Assume without loss of generality that $X^1(Q) \subset V$. In V the solutions of the linear flow can be given explicitly as in (4).

Write $J^c_t(z)$ for the absolute value of the determinant of the linear map $DX^t \mid F_z : F_z \to F_{X^t(z)}$ where z is any point of Λ and $t \in \mathbb{R}$.

For points z in $X^1(Q)$ and for $s > 0$ such that $X^t(z)$ remains in V for $0 \le t \le s$ we have

- $\big\|DX^t \mid E_z\big\| = e^{\lambda_2 t}$;
- $\big\|DX^t \mid E_z\big\| = e^{(\lambda_2-\lambda_3)t} \cdot m(DX^t \mid F_z)$;
- $|\det DX^t \mid F_z| = e^{(\lambda_1+\lambda_3)t}$,

where $m(\cdot)$ denotes the minimum norm of the linear map. Note that because $\lambda_1 + \lambda_3 > 0$ the flow in V expands volume along the F direction. Moreover since $\lambda_2 < \lambda_3$ the flow contracts along the E direction strongly than it expands along the F direction, by the second item above. We say that F *dominates* E. Observe that the above properties are also valid for the singularity p and the periodic orbit σ.

In what follows we extend these properties for the action of X on points of Λ for all times.

Notice that the flow takes a finite amount of time, bounded from above and from below, to take points in Q to $X^1(Q)$, and from $D^u(p)$ to Q (these times are constant and equal to 1 in our construction).

Hence if we are given a point $z \in \Lambda \setminus \{p, \sigma\}$, then its negative orbit $X^{-t}(z)$ for $t > 0$ will have consecutive and alternate hits on $D^u(p)$ and Q, at times $t_1 < s_1 < t_2 < s_2 < \cdots < t_n < s_n < \ldots$ respectively, with $t_0 = s_0 = 0$ and $r_n = |t_{n+1} - s_n|$ bounded from below by T_0 independently of $n \ge 1$.

Let $B > 0$ be a upper bound on $\|DX^{-t}(z)\|$ from 0 to T_1 and for all $z \in \Lambda$. Then from the volume expansion on V we have for $t_n < t \le s_n$

$$
\begin{aligned}
|\det DX^{-t} \mid F_z| &\le \exp\Big(B \cdot n - (\lambda_1+\lambda_3)\cdot\big(t - \sum_{i=1}^{n-1} r_i\big)\Big) \\
&= \exp\Big(t\cdot(\lambda_1+\lambda_3)\cdot\big(\frac{Bn}{t(\lambda_1+\lambda_3)} - 1 + \frac{\sum_{i=1}^{n-1} r_i}{t}\big)\Big).
\end{aligned}
$$

Since $t > T_0 n$ and $\sum_{i=1}^{n-1} r_i < t$ we see that there exists $K > 0$ such that $|\det DX^{-t} \mid F_z| \le K^{-1} \cdot e^{(\lambda_1+\lambda_3)t}$, which is equivalent to volume expansion.

The uniform contraction along E and the domination of F over E are obtained by similar arguments, see also Section 6.3.

6. Lorenz attractor and geometric models

Here we present a study of the Lorenz system of equations (1) and then explain the construction of the geometric Lorenz models, which initially where intended to mimic the behavior of the solutions of the system (1), but actually give an accurate description of this flow. Recall the relation between the Lorenz flow, and the associated geometrical model, with sensitive dependence on initial conditions and its historical impact, briefly touched upon in Section 2.

6.1. **Properties of the Lorenz system of equations.** Here we list analytical properties directly obtained from the Lorenz equations, which can be found with much more details in the books of Sparrow [5] and Guckenheimer-Holmes [6].

Let $X : \mathbb{R}^3 \to \mathbb{R}^3$ be the flow defined by the equations (1).

(1) *Singularities of* X. The origin $\sigma_0 = (0, 0, 0)$ is a singularity of the field X which does not depend on the parameters of X. The others are

$$\begin{aligned}&\sigma_1 = (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1) \quad \text{and}\\ &\sigma_2 = (\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1).\end{aligned}$$

(2) *Symmetry of* X. The map $(x, y, z) \mapsto (-x, -y, z)$ preserves the Lorenz system of equations, that is if $\big(x(t), y(t), z(t)\big)$ is a solution of the system of equations, then $\big(-x(t), -y(t), z(t)\big)$ will also be a solution.

(3) *Divergence of* X. We have

$$DX(x, y, z) = \begin{pmatrix} \partial_x(\dot{x}) & \partial_y(\dot{x}) & \partial_z(\dot{x}) \\ \partial_x(\dot{y}) & \partial_y(\dot{y}) & \partial_z(\dot{y}) \\ \partial_x(\dot{z}) & \partial_y(\dot{z}) & \partial_z(\dot{z}) \end{pmatrix} = \begin{pmatrix} -a & a & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

hence

$$\operatorname{div} X(x, y, z) = \nabla \cdot X = \operatorname{trace}\big(DX(x, y, z)\big) = -(a + 1 + b) < 0.$$

This shows the strongly dissipative character of this flow and implies that the flow contracts volume: if V_0 is the initial volume of a subset B of $\mathbb{R}^3$ we have by Liouville's Formula that the volume $V(t)$ of the image $X^t(B)$ is $V(t) = V_0 e^{-(\sigma+b+1)t}$. For the parameters in (1) we have $V(t) = V_0 e^{-\frac{41}{3}t}$.

In particular any maximally positively invariant subset under X^t has zero volume: Leb $\big(\cap_{t>0} X^t(U)\big) = 0$ for any open subset U of $\mathbb{R}^3$.

(4) *Eigenvalues of the singularities.* For the parameters in (1) the singularities are, besides σ_0

$$\sigma_1 = (-6\sqrt{2}, -6\sqrt{2}, 27) \quad \text{and} \quad \sigma_2 = (6\sqrt{2}, 6\sqrt{2}, 27).$$

For $DX(\sigma_0)$ we have the eigenvalues

$$\begin{aligned}\lambda_1 &= -11/2 + \sqrt{1201}/2 \approx 11.83;\\ \lambda_2 &= -11/2 - \sqrt{1201}/2 \approx -22.83;\\ \lambda_3 &= -8/3 \approx -2.67.\end{aligned}$$

Note that $-\lambda_2 > \lambda_1 > -\lambda_3 > 0$ which corresponds to a *Lorenz-like* singularity (Definition (3)).

For σ_1 the characteristic polynomial of $DX(\sigma_1)$ is of odd degree $p(x) = x^3 + \frac{41}{3}x^2 + \frac{304}{3}x + 1440$ and its derivative $p'(x) = 3x^2 + \frac{41}{4}x + \frac{304}{3}$ is strictly positive for all $x \in \mathbb{R}$, hence there exists a single real root λ of p. Since $p(0) > 0 > p(-15)$ the root is negative and simple numerical calculations show that $\lambda \approx -13.85457791$. Factoring p we get

$$\begin{aligned} p(x) &= \big(x - \lambda\big)\big(x^2 - 0.187911244x + 103.9367643\big) \\ &= \big(x - \lambda\big)\big(x - z\big)\big(x - \overline{z}\big) \end{aligned}$$

and thus $z \approx 0.093955622 + 10.19450522i$.

For σ_2 the eigenvalues are the same by the symmetry of X.

Using this we obtain the following Figure 16 of the local invariant manifolds and thus the local dynamics near the singularities

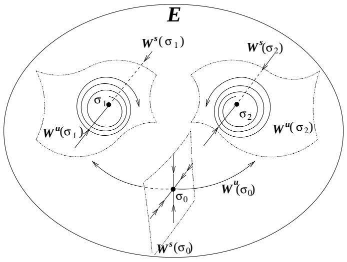

FIGURE 16. Local stable and unstable manifolds near σ_0, σ_1 and σ_2, and the ellipsoid E.

(5) *The trapping ellipsoid.* There exists an ellipsoid E where eventually every positive orbit of the flow enters. Moreover E is transverse to the flow X. Therefore the open region V bounded by E is a *trapping region* for X, that is $\overline{X^t(U)} \subset U$ for all $t > 0$.

This is obtained by finding an appropriate Lyapunov function. We follow Sparrow [29, Appendix C] (see also the original work of Lorenz [16]). Consider

$$L(x, y, z) = rx^2 + ay^2 + a(z - 2r)^2.$$

Then along solutions of the system (1) we have

$$\frac{dL}{dt} = -2a\big(rx^2 + y^2 + bz^2 - 2brz\big).$$

Let D be domain where $dL/dt \geq 0$ and let M be the maximum of L in D. Now define E to be the set of points such that $L \leq M + \varepsilon$ for some $\varepsilon > 0$ small. Since $D \subset E$ for x outside E we have $dL/dt = \nabla L \cdot X < -\delta < 0$ where $\delta = \delta(\varepsilon) > 0$ and X is the vector field defined by the equations (1). Then after a finite time the solution of

> the Lorenz system through x will enter the set E. Moreover for the values $(a, r, b) = (10, 28, 8/3)$ it is routine to check that ∇L points to the exterior of V over $\partial V = E$ and so all trajectories through E move towards the interior of V. Once in V any trajectory will remain there forever in the future.

Since $\overline{V}$ is compact the maximal positively invariant set $A = \cap_{t>0}\overline{X^t(V)}$ is an attracting set where trajectories of the flow accumulate when t grows without limit.

In fact numerical simulations show that there exists a subset B homeomorphic to a bi-torus such that every positive trajectory crosses B transversely and never leaves it. Hence the open set U bounded by B (see Figure 17) is a better candidate for the trapping region of the set with interesting limit dynamics for X, since σ_1 and σ_2 are isolated points in the ω-limit set of X. Hence $\Lambda = \cap_{t>0}\overline{X^t(U)}$ is also an attracting set and the origin is the only singularity contained in U.

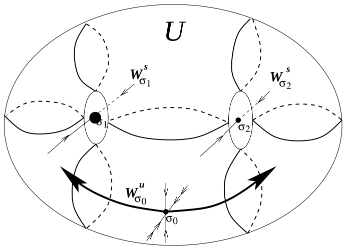

FIGURE 17. The trapping bi-torus.

6.1.1. *The evolution of a regular orbit inside the attracting basin.* Lorenz observed numerically what today is known as *sensitive dependence on initial conditions*. Due to this the actual path of any given orbit is impossible to calculate for all large values of integration time.

The "butterfly" which appear on the computer screens can be explained heuristically through the analytical properties already determined and by some numerical results. In fact the set of points whose orbits will draw the butterfly is the complement $\mathbb{R}^3 \setminus N$ of the union $N = W^s(\sigma_0) \cup W^s(\sigma_1) \cup W^s(\sigma_2)$ of the stable manifolds of the three singularities. Note that N is a bi-dimensional immersed surface in $\mathbb{R}^2$ and so has zero volume.

Figure 18 provides a very general view of how the orbit of a generic point in the trapping region U evolves in time. The trajectory starts spiraling around one of the singularities, σ_2 say, and suddenly "jumps" to the other singularity and then starts spiraling around σ_2. This process repeats endlessly. The number of turns around each singularity is essentially random. The ω-limit of a generic orbit is the following "butterfly" in Figure 19.

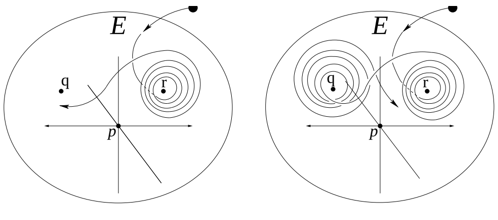

FIGURE 18. The evolution of a generic orbit inside U.

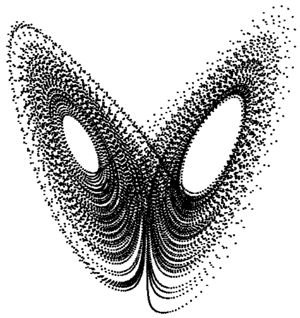

FIGURE 19. Another view of the Lorenz attractor.

6.2. **The geometric model.** The work of Lorenz on the famous flow was published in 1963 [16] but more than 10 years passed before new works on the subject appeared. Williams [35] wrote (in 1977):

> *. . . Several years ago Jim Yorke figured out some things about the Lorenz equation and got other mathematicians interested. He gave some talks on the subject, including one here at Berkeley. Ruelle, Lanford and Guckenheimer became interested and did some work on these equations. Unfortunately, except for the preprint of Ruelle, Guckhenheimer's paper, is the only thing these four people ever wrote on the subject as far as I know.*

Lorenz had already conjectured the existence of a strange attractor according to the available numerical simulations. The rigorous proof of this fact took many years due to the presence of a singularity accumulated by regular orbits of the flow, which prevents this set from being uniformly hyperbolic — see e.g. Section 3.

An important breakthrough in the understanding of the dynamics of the solutions of the Lorenz system of equations was achieved through the introduction of geometric models independently by Afraimovich, Bykov, Shil'nikov [1] in 1977 and by Guckenheimer, Williams [7] in 1979. These models were based on the properties suggested by the numerical simulations. In fact they were able to show the existence of a strange attractor for the geometric model.

This model inspired many others. Today there are different extensions and there are singular attractors which are not of the “Lorenz type”: neither conjugated nor equivalent to the Lorenz geometrical model, see e.g. [21].

As explained before, in 1998 a positive answer to the existence of a strange attractor in the original Lorenz system of equations was given by Tucker [31] in his PhD thesis, through the theory of normal forms together with rigorous numerical algorithms.

6.2.1. *Construction of the geometric model.* To present the detailed construction of the geometric Lorenz model we first analyze the dynamics in a neighborhood of the singularity at the origin, and then we imitate the effect of the pair of saddle singularities in the original Lorenz flow, as in Figure 16.

Near the singularity. By the Hartman-Grobman Theorem or by the results of Sternberg [30], in a neighborhood of the origin the Lorenz equations are equivalent to the linear system $(\dot{x}, \dot{y}, \dot{z}) = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$ through conjugation, thus

$$X^t(x_0, y_0, z_0) = (x_0 e^{\lambda_1 t}, y_0 e^{\lambda_2 t}, z_0 e^{\lambda_3 t}), \tag{4}$$

where $\lambda_1 \approx 11.83$, $\lambda_2 \approx -22.83$, $\lambda_3 = -8/3$ and $(x_0, y_0, z_0) \in \mathbb{R}^3$ is an arbitrary initial point near $(0, 0, 0)$.

Consider $S = \big\{(x, y, 1) : |x| \leq 1/2, \quad |y| \leq 1/2\big\}$ and

$$S^- = \big\{(x, y, 1) \in S : x < 0\big\}, \qquad S^+ = \big\{(x, y, 1) \in S : x > 0\big\} \quad \text{and}$$
$$S^* = S^- \cup S^+ = S \setminus \Gamma \qquad \text{, where} \quad \Gamma = \big\{(x, y, 1) \in S : x = 0\big\}.$$

Assume that S a transverse section to the flow so that every trajectory eventually crosses S in the direction of the negative z axis as in Figure 20. Consider also $\Sigma = \{(x, y, z) : |x| = 1\} = \Sigma^- \cup \Sigma^+$ with $\Sigma^\pm = \{(x, y, z) : x = \pm 1\}$. For each $(x_0, y_0, 1) \in S^*$ the time τ such that $X^\tau(x_0, y_0, 1) \in \Sigma$ is given by $\tau(x_0) = -\frac{1}{\lambda_1} \log |x_0|$, which depends on $x_0 \in S^*$ only and is such that $\tau(x_0) \to +\infty$ when $x_0 \to 0$. This is one of the reasons many

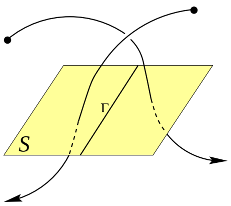

FIGURE 20. S is a cross-section of the flow.

standard numerical algorithms were unsuited to tackle the Lorenz system of equations. Hence we get (where $\operatorname{sgn}(x) = x/|x|$ for $x \neq 0$ as usual)

$$X^{\tau}(x_0, y_0, 1) = \big(\operatorname{sgn}(x_0), y_0 e^{\lambda_2 \tau}, e^{\lambda_3 \tau}\big) = \big(\operatorname{sgn}(x_0), y_0 |x_0|^{-\frac{\lambda_2}{\lambda_1}}, |x_0|^{-\frac{\lambda_3}{\lambda_1}}\big).$$

Since $0 < -\lambda_3 < \lambda_1 < -\lambda_2$, we have $0 < \alpha = -\frac{\lambda_3}{\lambda_1} < 1 < \beta = -\frac{\lambda_2}{\lambda_1}$. Let $L : S^* \to \Sigma$ be such that $L(x, y) = \big(y|x|^{\beta}, |x|^{\alpha}\big)$ with the convention that $L(x, y) \in \Sigma^+$ if $x > 0$ and $L(x, y) \in \Sigma^-$ if $x < 0$. It is easy to see that $L(S^{\pm})$ has the shape of a triangle without the vertex $(\pm 1, 0, 0)$. In fact these are cusp points of the boundary of each of these sets.

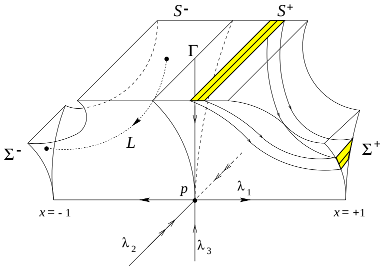

Figure 21. Behavior near the origin.

From now on we denote by $\Sigma^{\pm}$ the closure of $L(S^{\pm})$. Clearly each line segment $S^* \cap \{x = x_0\}$ is taken to another line segment $\Sigma \cap \{z = z_0\}$ as sketched in Figure 21.

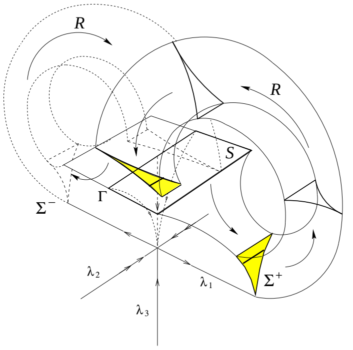

Figure 22. R takes $\Sigma^{\pm}$ to S.

The effect of the saddles. The sets $\Sigma^{\pm}$ should return to the cross section S through a rotation around $W^s(\sigma_1)$ and $W^s(\sigma_2)$. We assume that this rotation takes line segments $\Sigma \cap \{z = z_0\}$ into line segments $S \cap \{x = x_1\}$ as sketched in Figure 22.

We are assuming that the "triangles" $\Sigma^\pm$ are compressed in the y-direction and stretched on the other transverse direction. This is related to the eigenvalues of σ_1, σ_2 of the original Lorenz flow as explained below.

The rotation R mentioned above is assume to be such that for each $(y, z) \in \Sigma^\pm$

$$DR(y,z) = \begin{pmatrix} 0 & \pm M \\ \sigma & 0 \end{pmatrix} \quad \text{for some} \quad 0 < \sigma < 1 \quad \text{and} \quad M > 1,$$

and we define the Poincaré first return map $P : S^* \to S$ as $P = R \circ L$.

The combined effects of R and L on lines implies that the foliation of S given by the lines $S \cap \{x = x_0\}$ is invariant under the return map, meaning that for any given leaf γ of this foliation, its image $P(\gamma)$ is contained in a leaf of the same foliation. Hence P must have the form $P(x,y) = \big(f(x), g(x,y)\big)$ for some functions $f : I \setminus \{0\} \to I$ and $g : (I \setminus \{0\}) \times I \to I$, where $I = [-1/2, 1/2]$.

A consequence of all this is that every $x \in S$ has a positive orbit disjoint from $W^{ss}(\sigma)$. Since every point $x \in \Lambda \setminus \{\sigma\}$ has a positive orbit that will eventually cross S by construction, we see that

$$W^{ss}(\sigma) \cap \Lambda = \{\sigma\}. \tag{5}$$

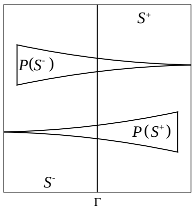

Figure 23. $P(S^*)$.

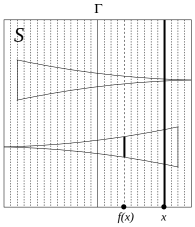

Figure 24. Projection on I.

Properties of the one-dimensional map f. Now we specify the properties which we impose on f:

(f1) the symmetry of the Lorenz equations implies $f(-x) = -f(x)$.

(f2) f is discontinuous at $x = 0$ with lateral limits $f(0^-) = +\frac{1}{2}$ and $f(0^+) = -\frac{1}{2}$, since P is not defined at Γ because $\Gamma \subset W^s(0,0,0)$.

(f3) f is differentiable on $I \setminus \{0\}$ and $f'(x) > \sqrt{2}$, since the real part of the (complex) eigenvalues of the saddles σ_1, σ_2 is positive (see the previous Section 6.1).

(f4) the lateral limits of f' at $x = 0$ are $f'(0^-) = +\infty$ and $f'(0^+) = -\infty$.

On the other hand $g : S^* \to I$ is defined in such a way that it contracts the second coordinate: we assume $g'_y(w) \le \mu < 1$ for all $w \in S^*$. This is suggested by the eigenvalues $\lambda_2 \approx -22.83$ of σ_0 and $\lambda \approx -13.8545$ of the saddles σ_1, σ_2 (see Section 6.1). Moreover the rate of contraction of g on the second coordinate should be much higher than the expansion rate of f. Figure 23 sketches $P(S^*)$. In addition the expansion rate is big enough to obtain a strong mixing property for f (it is locally eventually onto, see Section 6.5).

The foliation is contracting in the following sense: for a given leaf γ of the foliation and for $x, y \in \gamma$ then dist $\big(P^n(x), P^n(y)\big) \to 0$ when $n \to \infty$.

Thus the study of the 3-flow can be reduced to the study of a bi-dimensional map and, moreover, the dynamics of this map can be further reduced to a one-dimensional map, since the invariant contracting foliation enables us to identify two points on the same leaf, since their orbits remains forever on the same leaf and the distance of their images tends to zero under iteration, see Figure 24 for a sketch of this identification.

The quotient map obtained through this identification will be called *the Lorenz map.*

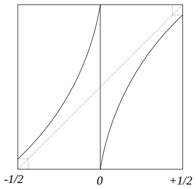

FIGURE 25. The Lorenz map f.

6.3. **The geometric Lorenz attractor is a partially hyperbolic set with volume expanding central direction.** Observe that the time $t(w)$ it takes a point $w \in \Sigma$ to go to S, that is $X^{t(w)}(w) \in S$ and $X^t(w) \in \mathbb{R}^3 \setminus (S \cup \Sigma)$ for $0 < t < t(w)$, is bounded by some constant independently of the point: $t(w) \le t_0$. This ensures that the behavior of the flow on the maximal positively invariant subset of the trapping region is prescribed by the behavior from the cross-section S to the cross-sections Σ^+, Σ^-, as we now explain.

Figure 21 makes it clear that the linear flow (4) preserves lines in the direction of the y-axis when taking points from S to Σ. Moreover it is not difficult to check that *its derivative* DX^t also preserves planes orthogonal to the y-axis.

In addition, by the choice of the flow from Σ to S and as Figure 22 suggests, horizontal lines at Σ, i.e. parallel to the y-axis, are taken to lines parallel to same axis in S, that is *the flow preserves lines parallel to the y-axis from Σ to S. Since the flow from Σ to S is essentially a rotation, we can assume that its derivative also preserves planes orthogonal to the y-axis.*

From this we deduce that the following splitting of $\mathbb{R}^3$: $E = \mathbb{R} \times \{(0,0\}$ and $F = \{0\} \times \mathbb{R}^2$, is preserved by the flows, that is $DX^t_w \cdot E = E$ and $DX^t_w \cdot F = F$ for all t and every point w in an orbit inside the trapping ellipsoid.

Moreover we can check that for w on the linearised part of the flow, from S to Σ, we have for $t > 0$ such that $X^{[0,t]}(w)$ is contained in the domain of the linearised coordinates:

- $\|DX^t_w \mid E\| = e^{\lambda_2 t}$;
- $\|DX^t_w \mid E\| = e^{(\lambda_2-\lambda_3)t} \cdot m\big(DX^t \mid F\big)$,

where $m\big(DX^t \mid F\big)$ is the minimum norm of the linear map. Since $\lambda_2 < 0$ we see that E is uniformly contracting, this a stable direction. But $\lambda_2 - \lambda_3 < 0$ and so the contraction along the direction of F is weaker than the contraction along E. This kind of splitting $E \oplus F$ of $\mathbb{R}^3$ is called a *partially hyperbolic splitting.*

Observe also that since $\lambda_1+\lambda_3 > 0$ we have that $|\det DX^t \mid F| = e^{(\lambda_1+\lambda_3)t}$ and so the flow *expands volume* along the F direction.

We will see that these properties characterize compact invariant sets which are robustly transitive.

However we have only checked these properties in the linearised region. But if the orbit of a point w passes outside the linear region k times from Σ to S lasting $s_1 + \cdots + s_k$ from time 0 to time t, then $t > s_1 + \cdots + s_k$ and for some constant $b > 0$ bounding the derivatives of DX^t from 0 to t_0 we have

$$\|DX^t_w \mid E\| \le e^{bk+\lambda_2(t-s_1-\cdots-s_k)} = \exp\Big\{\lambda_2 t\big(1 - \frac{bk}{\lambda_2 t} - \frac{s_1+\cdots+s_k}{t}\big)\Big\},$$

so the last expression in brackets is bounded. We see that E is (K,λ_2)-contracting for some $K > 0$.

An entirely analogous reasoning shows that the direction E dominates F uniformly for all t and that DX^t expands volume along F also uniformly.

Thus the maximal positively invariant set in the trapping ellipsoid is partially hyperbolic and the flow expands volume along a bi-dimensional invariant direction.

6.4. **Existence and robustness of invariant stable foliation.** Now we prove, partially following the work of [7], that the geometric Lorenz attractor constructed in the previous subsection is *robust*, that is, it persists for all nearby vector fields.

More precisely: there exists a neighborhood U in $\mathbb{R}^3$ containing the attracting set Λ such that for all vector fields Y which are C^1-close to X the maximal invariant subset in U, that is $\Lambda_Y = \cap_{t\ge 0} Y^t(U)$, is still a transitive Y-invariant set.

This is a striking property of these flows since the Lorenz flow exhibits sensitive dependence on initial conditions. The robustness will be a consequence of the persistence of the invariant contracting foliation on the cross-section S to the flow.

Numerically this is expected since in spite of the huge integration errors involved and the various integration algorithms used the solutions obtained always have a shape similar to the one in Figure 19, independently of the initial point chosen to start the integration.

We start by obtaining the persistence of the stable foliation for points in the attractor, then explain why these attractors, although robust, are *not structurally stable*, in Section 6.5.3.

6.4.1. *Geometric idea of the proof.*

Theorem 6.1 (Persistence of contracting foliation)**.** *Let X be the vector field obtained in the construction of the geometric Lorenz model and $\mathcal{F}_X$ the invariant contracting foliation of the cross-section S. Then any vector field Y which is sufficiently C^1-close to X admits an invariant contracting foliation $\mathcal{F}_Y$ on the cross-section S.*

We first present a geometric idea of the proof and then proceed to the details. Observe first that the cross-section S remains transverse to any flow C^1-close to X and that the singularities $\sigma_0, \sigma_1, \sigma_2$ persist with eigenvalues satisfying the same relations as before since they are hyperbolic. In addition, since $W^u_X(\sigma_0)$ intersects S transversely, then just by the C^1 continuous variation of compact parts of the unstable manifolds of a hyperbolic singularity we have that $W^u_Y(\sigma_0(Y))$ still intersects S transversely for all Y close to X in the C^1 norm.

Without loss we can assume, after a C^1 change of coordinates, that the Lorenz-like singularity $\sigma_0(Y)$ remains at the origin and that the eigenvectors of $DY(\sigma_0(Y))$ have the directions of the coordinate axis as before, with the plane $x = 0$ containing the stable manifold of $\sigma_0(Y)$.

Thus for a neighborhood $\mathcal{U}$ of X in the C^1 topology and for each $Y \in \mathcal{U}$ we can define the Poincaré first return map $P_Y : S^* \to S$ as $P_Y = R_Y \circ L_Y$ where $L_Y : S^* \to \Sigma$ is such that $L_Y(x, y) = (y|x|^\beta, |x|^\alpha)$ with $\alpha = -\frac{\lambda_3(Y)}{\lambda_1(Y)}$ and $\beta = -\frac{\lambda_2(Y)}{\lambda_1(Y)}$ (note that $\beta - \alpha > 1$).

On the other hand $R_Y : \Sigma \to S$ is a C^1-diffeomorphism which can be expressed by the composition $R_Y = J_Y \circ R_0$, where J_Y is a C^1-perturbation of the identity and R_0 is the diffeomorphism associated to X_0.

Now let $\mathcal{A}$ be the space of continuous maps $\phi : \mathcal{U} \times S \to [-1, +1]$. For each $Y \in \mathcal{U}$ and $\phi \in \mathcal{A}$ we define $\phi_Y : S \to [-1, 1]$ by $\phi_Y(q) = \phi(Y, q)$ for all $q \in S$. We associate to ϕ_Y a vector field $\eta^\phi_Y : S \to [-1, 1] \times \{1\}$ given by $\eta^\phi_Y(q) = \big(\phi_Y(q), 1\big)$ which we view as a vector on $T_qS = \mathbb{R}^2$. Integrating the field η^ϕ_Y we get a family of curves which induces a foliation on S. We must

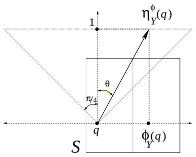

FIGURE 26. The field η^ϕ_Y.

show that there exists $\phi \in \mathcal{A}$ such that η_Y^ϕ induces an invariant foliation under P_Y. Before explaining the proof of this fact we state a necessary and sufficient condition for the invariance of this foliation.

Let F be a continuous vector field defined on S and P the map defined above. Integrating F we get a foliation of S. Let $q \in S^*$ have image $P(q)$ and consider the vectors $F(q)$ and $F\big(P(q)\big)$. Observe that the foliation induced by F is invariant under P if

$$DP(q)\big(F(q)\big) \quad \text{and} \quad F\big(P(q)\big) \quad \text{are parallel,} \quad \text{or}$$
$$F(q) \quad \text{and} \quad [DP(q)]^{-1}F\big(P(q)\big) \quad \text{are parallel.}$$

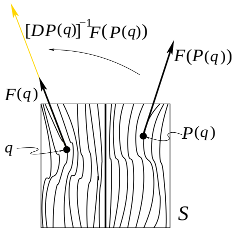

Figure 27. The field F and the parallel condition.

On the other hand if we consider the slope of vectors with respect to the vertical direction $(0, 1)$, two vectors are parallel if, and only if, their slope is the same. For $(a, b) \in \mathbb{R}^2$ we set slope $(a, b) = a/b$ and hence to check that the foliation defined by F is invariant under P amounts to obtain

$$\text{slope}\,\big(F(q)\big) = \text{slope}\,\Big([DP(q)]^{-1}F\big(P(q)\big)\Big).$$

Translating this for η_Y^ϕ we obtain the condition

$$\phi_Y(q) = \text{slope}\,\Big(\eta_Y^\phi(q)\Big) = \text{slope}\,\Big(\big[DP_Y(q)\big]^{-1}\eta_Y^\phi\big(P_Y(q)\big)\Big).$$

The last term above depends on ϕ, X and q and if we define $T : \mathcal{A} \to \mathcal{A}$ as

$$T\big(\phi_Y\big)(q) = \text{slope}\,\Big(\big[DP_Y(q)\big]^{-1}\eta_Y^\phi\big(P_Y(q)\big)\Big),$$

then the condition of invariance becomes $T\big(\phi_Y\big)(q) = \phi_Y(q)$, that is $T(\phi) = \phi$. Hence the element $\phi \in \mathcal{A}$ for which η_X^ϕ induces and invariant foliation on S is a fixed point of T. Thus we are left to prove that the operator T has a fixed point.

For this, we first show that T is well defined and then that T is a contraction in an appropriate space, which concludes the proof of Theorem 6.1.

6.5. **Robustness of the geometric Lorenz attractors.** Here we conclude the proof that the geometric Lorenz attractor is a robustly transitive attractor and show that it is not structurally stable. Here we drop condition (f1) on the symmetry of the one-dimensional map f.

6.5.1. *Robust properties of the one-dimensional map f.* We start by showing that the properties of the one-dimensional map f are robust for small C^1 perturbations of X.

Indeed, note that since the stable foliation is robust, we can define the one-dimensional map f_Y as the quotient map of the corresponding Poincarï¿½ map P_Y over the leaves of the foliation $\mathcal{F}_Y$, for all flows Y close to X in the C^1 topology.

Moreover since the leaves of $\mathcal{F}_Y$ are C^1 close to those of $\mathcal{F}$, hence f_Y is C^1 close to f and thus there exists $c \in [-1/2, 1/2]$ which play for f_Y the same role of 0 so that properties (f2)-(f4) from Section 6.2.1 are still valid for f_Y on a subinterval $[-b, b]$ for some $0 < b < 1/2$ close to $1/2$.

This implies that every f_Y is *locally eventually onto* for all Y close to X, that is for any interval $J \subset (-b, b)$ there exists an iterate $n \geq 1$ such that $f_Y^n(J) = (-b, b)$.

Lemma 6.2. *Let $f : [-1/2, 1/2] \setminus \{0\} \to [-1/2, 1/2]$ be given satisfying the properties (f1)-(f4). Then f is locally eventually onto: for any open interval J not containing 0 there exists n such that $f^n \mid J$ is a diffeomorphism between J and $(f(-1/2), f(1/2))$.*

This implies in particular the maps f_Y are (robustly) transitive and periodic points are dense. Moreover this also implies that the pre-orbit set $\cup_{n\geq 0} f^{-n}\{x\}$ is dense for every $x \neq 0$.

Proof. Let $J_0 \subset (-1/2, 1/2)$ be an open interval with $0 \notin J_0$ and let $\eta = \inf |f'| > \sqrt{2}$.

Since $0 \notin J_0$ then $f(J_0)$ is such that $\ell\big(f(J_0)\big) \geq \eta\ell(J_0)$, where $\ell(\cdot)$ denotes length, and $f(J_0)$ is connected.

1. If $0 \notin f(J_0)$, set $J_1 = f^2(J_0)$ and then $\ell(J_1) \geq \eta^2\ell(J_0)$.
2. If $0 \in f(J_0)$, then $f^2(J_0) = I^- \cup I^+$, where I^+ is the biggest connected component. Thus
$$\ell(I^+) \geq \frac{\ell\big(f^2(J_0)\big)}{2} \geq \frac{\eta^2}{2}\ell(J_0).$$

Now replace J_0 by I^+ in case (2) or by J_1 in case (1). Since $\min\{\eta, \eta^2/2\} > 1$ we obtain after finitely many steps one of the intervals $(-1/2, 0)$ or $(0, 1/2)$. One more interate then covers the interval $(f(-1/2), f(1/2))$.

□

6.5.2. *Transitivity and denseness of periodic orbits.* We deduce these features from a stronger property: we show that the geometric Lorenz attractor is a homoclinic class.

Proposition 6.3. *There exists a periodic orbit $\mathcal{O}_X(p)$ in the geometric Lorenz attractor Λ such that $\Lambda = H_X(p) = \overline{W_X^s(p) \pitchfork W_X^u(p)}$.*

We prove this in Section 6.6. Observe that *every periodic orbit $\mathcal{O}(p)$ in Λ must be hyperbolic* since

- the uniformly contracting foliation obtained in Section 6.4 provides a uniformly contracting direction and a stable manifold for $\mathcal{O}(p)$: if $\mathcal{F}(p)$ is the leaf of $\mathcal{F}$ through $p = \mathcal{O}(p) \cap S$, then
$$W^s\big(\mathcal{O}(p)\big) = \bigcup_{t \geq 0} X^{-t}\big(\mathcal{F}(p)\big);$$
- the expansion of the one-dimensional projection map f ensures that there exists a forward DP-invariant expanding cone field around the horizontal direction, which in turn ensures the existence of a DP-invariant expanding direction at p.

Proposition 6.3 implies after the Birkhoff-Smale that the geometric Lorenz attractor Λ has a dense orbit and a dense subset of periodic orbits.

Since the arguments we use to prove Proposition 6.3 depend only on the properties of f and these properties are robust, we conclude that the geometric Lorenz attractors are robustly transitive.

6.5.3. *The geometric Lorenz models are not structurally stable.* The dynamics of two nearby geometric Lorenz models are in general not topologically equivalent. In fact Guckenheimer and Williams [7, 36] show that the conjugacy classes are completely described by two parameters: the *kneading sequences* of the two singular values
$$f(0^+) = \lim_{x \to 0^+} f(x) \quad \text{and} \quad f(0^-) = \lim_{x \to 0^-} f(x)$$
with respect to the singular point 0 — *a pair of one-dimensional Lorenz-like maps are conjugate if, and only if, they have the same pair of kneading sequences* and, moreover, the corresponding flows are topologically equivalent if, and only if, the one-dimensional maps are conjugated (recall that we have dropped condition (f1)).

The kneading sequence of $x^+ = f(0^+)$ with respect to 0 is a sequence defined by
$$a_n = \begin{cases} 0 & \text{if } f^n(x^+) < 0 \\ 1 & \text{otherwise} \end{cases} ; \quad \text{for} \quad n \geq 0,$$
and analogously we define the kneading sequence $(b_n)_{n \geq 0}$ for x^-.

It is easy to see that *if two nearby geometric Lorenz flows are topologically conjugated then the kneading sequences must be equal*, since the equivalence relation preserves the orbit structure and in particular preserves also the first return iterates to the cross-section S.

Now given a geometric Lorenz flow X with corresponding kneading sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$, we can through a small perturbation find a C^1 close vector field Y whose corresponding one-dimensional map has kneading sequences $(a'_n)_{n \geq 0}$ and $(b'_n)_{n \geq 0}$ distinct from the pair $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$.

Indeed, if one of the orbits of $x^{\pm}$ is dense in $(-1, 1)$, then one of its iterate is arbitrarily close to 0. Thus a small perturbation of the map will flip one of the elements of the kneading sequence from 0 to 1 or viceversa. Otherwise there exists $\varepsilon > 0$ such that the orbits of $x^{\pm}$ do not enter $(-\varepsilon, \varepsilon)$. As we have already proved, the one-dimensional map f is locally eventually onto and in particular topologically transitive. Hence there exists a point $0 < y < \delta \ll \varepsilon$ with $0 < f(y) - x^{+} < \delta$ whose orbit is dense. Let $n > 0$ be the smallest integer such that $|f^n(y)| < \delta$. Consider $\tilde{f}$ a small perturbation of f such that

- $\tilde{f}$ satisfies all the properties (f1) through (f4);
- $\tilde{f} \mid [-1,1] \setminus (0,\delta) \equiv f$;
- $\tilde{f}(0^+) = y$.

Then $\tilde{f}^k\big(\tilde{f}(0^+)\big) = f^k(y)$ for $k = 0, \dots, n$ and so $\tilde{f}^n(y) \in (-\delta, \delta)$. Now we can perturb $\tilde{f}$ so that $\tilde{f}^n(y)$ changes sign and this would change one of the kneading sequences of $\tilde{f}$. Since δ can be taken arbitrarily small, then we obtain a very small perturbation of f whose kneading sequences are distinct. Since we can build a geometric Lorenz flow from $\tilde{f}$ and from any of its small perturbations, we have shown that we can always find a nearby geometric Lorenz flow Y not topologically conjugated to the given X.

6.6. **The geometric Lorenz attractor is a homoclinic class.** Here we prove Proposition 6.3 following Bautista [3].

Observe first that the *geometric Lorenz attractor* Λ *must contain a hyperbolic periodic orbit.* Indeed since the associated Lorenz transformation f is locally eventually onto, the periodic orbits of f are dense. Let $x_0, \dots, x_k$ be a periodic orbit of f. Then the leaves $\ell_0, \dots, \ell_k$ of $\mathcal{F}$ in S which project on these points form a invariant set under the map P. Since P preserves the leaves of the foliation $\mathcal{F}$ and is a contraction along $\mathcal{F}$, then P^k must send each ℓ_i into itself with a uniform contraction rate. Hence there exists a point p_i which is fixed by P^k on each leaf ℓ_i, i.e. $p_0, \dots, p_k$ is a periodic orbit of P.

The definition of P shows that the orbit of p_0 by the flow X is periodic and $\mathcal{O}_X(p_0) \cap S = \{p_0, \dots, p_k\}$.

As already observed every periodic orbit in Λ must be hyperbolic of saddle-type: the expanding and contracting directions can be easily read from the discussion in Section 6.3. Hence the unstable manifold of p is a disk transverse to S which intersects S in a one-dimensional manifold. The connected component of $W^u(p) \cap S$ which contains p is then a small line transverse to the foliation $\mathcal{F}$.

Now observe that since Λ is an attracting set, that is $\Lambda = \cap_{t>0} \overline{X^t(U)}$, where U is the trapping ellipsoid, then the unstable manifold $W^u(p)$ of the orbit of $p = p_0$ must be contained in Λ. Indeed if $z \in W^{uu}(p)$ then $\operatorname{dist}(X^{-t}(z), X^{-t}(p)) \xrightarrow[t \to +\infty]{} 0$ and hence $X^{-t}(z) \in U$ for big $t > 0$, thus $z \in X^t(U)$. This shows that $W^{uu}(p) \subset \Lambda$ and since Λ is X-invariant we also get $W^u(p) \subset \Lambda$.

The definition of homoclinic class and the fact that Λ is closed imply that $H_X(p) \subset \Lambda$. For the converse we need a stronger fact.

Lemma 6.4. *If Λ is the geometric Lorenz attractor and $p \in \Lambda$ is the point of some periodic orbit, then $\Lambda = \overline{W^u(p)}$.*

Proof. Let $w \in \Lambda \setminus \{\sigma\}$ be given. Then there exists $t \geq 0$ such that $y = X^t(w) \in S$. Let $\ell = \mathcal{F}(y)$ be the corresponding leaf of $\mathcal{F}$ through y. Then ℓ is not the leaf $S \setminus S^*$. Therefore it projects to a point $x \in (-1/2, 0) \cup (0, 1/2)$. Since the pre-orbit set of every point is dense (because f is locally eventually onto), by definition of f this implies that $\Lambda \cap S = \Lambda \cap \overline{\cup_{n\geq 0} P^{-n}\ell}$.

Hence we have that $P^{-n}\ell \cap W^u(p) \neq \emptyset$ for some $n \geq 0$. But this means that $W^s(y) \cap W^u(p) \neq \emptyset$ and so $w, y \in \overline{W^u(p)}$. Thus $\Lambda \setminus \{\sigma\} \subset \overline{W^u(p)}$. $\square$

Finally to prove that $\Lambda \subset H_X(p)$ it is enough to show that $W^u(p) \subset H_X(p)$. Every point $w \in W^u(p)$ admits $t < 0$ such that $q = X^t(w) \in S$. Take a small neighborhood J of q in $W^u(p) \cap S$, which is a small line transverse to $\mathcal{F}$.

Let l be the leaf of $\mathcal{F}$ containing p and let I be the interval inside $(-1/2, 1/2)$ corresponding to J by the projection $S \to S/\mathcal{F} = (-1/2, 1/2)$. Recall that $l \subset W^s(p) \cap S$. Write x for the point corresponding to p under this projection.

Again by Lemma 6.2 there exists $n \geq 0$ such that $f^{-n}\{x\} \cap I \neq \emptyset$. This means that $J \pitchfork P^{-n}(l) \neq \emptyset$, hence in J there exists a point of the homoclinic class of p. Since J can be taken arbitrarily small near q, we conclude that $q \in H_X(p)$. This concludes the proof that $\Lambda = H_X(p)$.

7. GLOBAL DYNAMICS OF GENERIC 3-FLOWS

A compact invariant set Λ of X is *partial hyperbolic* if there are a continuous invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ and constants $\lambda, K > 0$ such that

- E^s_Λ (K, λ)-*dominates* E^c_Λ, i.e. for all $x \in \Lambda$ and for all $t \geq 0$
$$\|DX^t(x) \mid E^s_x\| \leq \frac{e^{-\lambda t}}{K} \cdot m(DX^t(x) \mid E^c_x);$$
- E^s_Λ is (K, λ)-contracting (see Section 3).

We shall say that $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ is a (K, λ)-*splitting* for short. For $x \in \Lambda$ and $t \in \mathbb{R}$ we let $J^c_t(x)$ be the absolute value of the determinant of the linear map $DX^t(x) \mid E^c_x : E^c_x \to E^c_{X^t(x)}$. We say that the sub-bundle E^c_Λ of the partial hyperbolic set Λ is *volume expanding* if

$$J^c_t(x) = \left|\det(DX^t \mid E^c_x)\right| \geq K e^{\lambda t},$$

for every $x \in \Lambda$ and $t \geq 0$ (in this case we say that E^c_Λ is (K, λ)-*volume expanding* to indicate the dependence on (K, λ).

It is known (see e.g. [22]) that a non-singular partially hyperbolic set for a three-dimensional flow, with volume expanding central direction, is uniformly hyperbolic.

Definition 2. A partial hyperbolic set is *singular-hyperbolic* if its singularities are hyperbolic and it has volume expanding central direction.

A *singular-hyperbolic attractor* is a singular-hyperbolic set which is an attractor as well: an example is the (geometric) Lorenz attractor presented in Section 6. A *singular-hyperbolic repeller* of X is a singular-hyperbolic attractor of $-X$. An example of a singular-hyperbolic set which is neither an attractor or a repeller is the singular horseshoe presented in Section 5.

The following result characterizes robust attractors for three-dimensional flows.

Theorem 7.1. *Robust attractors of* $X \in \mathfrak{X}^1(M^3)$ *containing singularities are singular-hyperbolic sets for* X*.*

Note that robust attractors cannot be C^1 approximated by vector fields presenting either attracting or repelling periodic points. This implies that, on 3-manifolds, any periodic point lying in a robust attractor is hyperbolic of saddle-type. Thus, as in Liao [15, Theorem A], we conclude that robust attractors *without singularities* on closed 3-manifolds are hyperbolic. Therefore we obtain a dichotomy as follows.

Theorem 7.2. *Let* Λ *be a robust attractor of* $X \in \mathfrak{X}^1(M)$*. Then* Λ *is either hyperbolic or singular-hyperbolic.*

Since the singularities σ in our setting are all Lorenz-like, the unstable manifold $W^u(\sigma)$ is one-dimensional, and there is a one-dimensional strong-stable manifold $W^{ss}(\sigma)$ contained in the two-dimensional stable manifold $W^s(\sigma)$. Using the linearization given by the Hartman-Grobman Theorem orbits of the flow in a small neighborhood U_0 of the singularity are solutions of the linear system (4), modulo a continuous change of coordinates.

Then for some $\delta > 0$ we may choose cross-sections contained in U_0

- $\Sigma^{o,\pm}$ at points $y^\pm$ in different components of $W^u_{loc}(\sigma) \setminus \{\sigma\}$
- $\Sigma^{i,\pm}$ at points $x^\pm$ in different components of $W^s_{loc}(\sigma) \setminus W^{ss}_{loc}(\sigma)$

and Poincaré first hitting time maps $R^\pm : \Sigma^{i,\pm} \setminus \ell^\pm \to \Sigma^{o,-} \cup \Sigma^{o,+}$, where $\ell^\pm = \Sigma^{i,\pm} \cap W^s_{loc}(\sigma)$, satisfying (see Figure 28)

1. every orbit in the attractor passing through a small neighborhood of the singularity σ intersects some of the incoming cross-sections $\Sigma^{i,\pm}$;
2. $R^\pm$ maps each connected component of $\Sigma^{i,\pm} \setminus \ell^\pm$ diffeomorphically inside a different outgoing cross-section $\Sigma^{o,\pm}$, preserving the corresponding stable foliations.

These cross-sections may be chosen to be planar relative to some linearizing system of coordinates near σ e.g. for a small $\delta > 0$

$$\Sigma^{i,\pm} = \{(x_1, x_2, \pm 1) : |x_1| \le \delta, |x_2| \le \delta\} \quad \text{and}$$
$$\Sigma^{o,\pm} = \{(\pm 1, x_2, x_3) : |x_2| \le \delta, |x_3| \le \delta\},$$

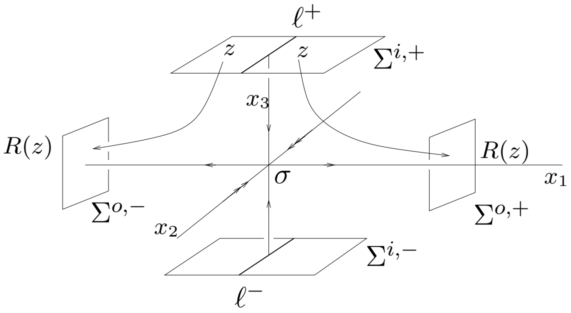

Figure 28. Cross-sections near a Lorenz-like singularity.

where the x_1-axis corresponds to the unstable manifold near σ, the x_2-axis to the strong-stable manifold and the x_3-axis to the weak-stable manifold of the singularity which, in turn, is at the origin, see Figure 28.

The singularity is hyperbolic for the vector field X. Hence for every C^1 nearby vector field Y there exists a unique Lorenz-like singularity σ_Y in U_0. Moreover the submanifolds $\Sigma^{i,\pm}$ and $\Sigma^{o,\pm}$ remain transverse to Y. So all local properties of these cross-sections are robust under small C^1 perturbations of the flow.

8. Cross-sections and Poincaré maps

For future reference we give here a few properties of *Poincaré maps*, that is, continuous maps $R : \Sigma \to \Sigma'$ of the form $R(x) = X^{t(x)}(x)$ between cross-sections Σ and Σ' of the flow near a singular-hyperbolic set. We always assume that the Poincaré time $t(\cdot)$ is large enough as explained in what follows.

8.1. **Hyperbolicity of Poincaré maps.**

Let Σ be a small cross-section to X and let $R : \Sigma \to \Sigma'$ be a Poincaré map $R(y) = X^{t(y)}(y)$ to another cross-section Σ' (possibly $\Sigma = \Sigma'$). Note that R needs not correspond to the first time the orbits of Σ encounter Σ'.

The splitting $E^s \oplus E^{cu}$ over U_0 induces a continuous splitting $E^s_\Sigma \oplus E^{cu}_\Sigma$ of the tangent bundle $T\Sigma$ to Σ (and analogously for Σ'), defined by

$$E^s_\Sigma(y) = E^{cs}_y \cap T_y\Sigma \quad \text{and} \quad E^{cu}_\Sigma(y) = E^{cu}_y \cap T_y\Sigma. \tag{6}$$

We now show that if the Poincaré time $t(x)$ is sufficiently large then (6) defines a hyperbolic splitting for the transformation R on the cross-sections, at least restricted to Λ, in the following sense.

Proposition 8.1. *Let $R : \Sigma \to \Sigma'$ be a Poincaré map as before with Poincaré time $t(\cdot)$. Then $DR_x(E^s_\Sigma(x)) = E^s_\Sigma(R(x))$ at every $x \in \Sigma$ and $DR_x(E^{cu}_\Sigma(x)) = E^{cu}_\Sigma(R(x))$ at every $x \in \Lambda \cap \Sigma$.*

Moreover for every given $0 < \lambda < 1$ there exists $t_1 = t_1(\Sigma, \Sigma', \lambda) > 0$ such that if $t(\cdot) > t_1$ at every point, then

$$\|DR \mid E^s_\Sigma(x)\| < \lambda \quad \textit{and} \quad \|DR \mid E^{cu}_\Sigma(x)\| > 1/\lambda \quad \textit{at every } x \in \Sigma.$$

Remark 8.2. In what follows we use K as a generic notation for large constants depending only on a lower bound for the angles between the cross-sections and the flow direction, and on upper and lower bounds for the norm of the vector field on the cross-sections. The conditions on t_1 in the proof of the proposition depend only on these bounds as well. In all our applications, all these angles and norms will be uniformly bounded from zero and infinity, and so both K and t_1 may be chosen uniformly.

Proof. The differential of the Poincaré map at any point $x \in \Sigma$ is given by

$$DR(x) = P_{R(x)} \circ DX^{t(x)} \mid T_x\Sigma,$$

where $P_{R(x)}$ is the projection onto $T_{R(x)}\Sigma'$ along the direction of $X(R(x))$. Note that $E^s_\Sigma(x)$ is tangent to $\Sigma \cap W^s(x) \supset W^s(x, \Sigma)$. Since the stable manifold $W^s(x)$ is invariant, we have invariance of the stable bundle:

$$DR(x)\big(E^s_\Sigma(x)\big) = E^s_{\Sigma'}\big(R(x)\big).$$

Moreover for all $x \in \Lambda$ we have

$$DX^{t(x)}\big(E^{cu}_\Sigma(x)\big) \subset DX^{t(x)}\big(E^{cu}_x\big) = E^{cu}_{R(x)}\,.$$

As $P_{R(x)}$ is the projection along the vector field, it sends $E^{cu}_{R(x)}$ to $E^{cu}_{\Sigma'}(R(x))$. This proves that the center-unstable bundle is invariant restricted to Λ, i.e. $DR(x)\big(E^{cu}_\Sigma(x)\big) = E^{cu}_{\Sigma'}(R(x))$.

Next we prove the expansion and contraction statements. We start by noting that $\|P_{R(x)}\| \le K$. Then we consider the basis $\{\frac{X(x)}{\|X(x)\|},\, e^u_x\}$ of E^{cu}_x, where e^u_x is a unit vector in the direction of $E^{cu}_\Sigma(x)$. Since the flow direction is invariant, the matrix of $DX^t \mid E^{cu}_x$ relative to this basis is upper triangular:

$$DX^{t(x)} \mid E^{cu}_x = \begin{bmatrix} \frac{\|X(R(x))\|}{\|X(x)\|} & \star \\ 0 & \Delta \end{bmatrix}.$$

Moreover

$$\frac{1}{K} \cdot \det\big(DX^{t(x)} \mid E^{cu}_x\big) \le \frac{\|X(R(x))\|}{\|X(x)\|}\Delta \le K \cdot \det\big(DX^{t(x)} \mid E^{cu}_x\big).$$

Then

$$\begin{aligned} \|DR(x)\, e^u_x\| &= \|P_{R(x)}\big(DX^{t(x)}(x) \cdot e^u_x\big)\| = \|\Delta \cdot e^u_{R(x)}\| = |\Delta| \\ &\ge K^{-3}\,|\det(DX^{t(x)} \mid E^{cu}_x)| \ge K^{-3}\lambda^{-t(x)} \ge K^{-3}\,\lambda^{-t_1}. \end{aligned}$$

Taking t_1 large enough we ensure that the latter expression is larger than $1/\lambda$.

To prove $\|DR \mid E^s_\Sigma(x)\| < \lambda$, let us consider unit vectors $e^s_x \in E^s_x$ and $\hat{e}^s_x \in E^s_\Sigma(x)$, and write

$$e^s_x = a_x \cdot \hat{e}^s_x + b_x \cdot \frac{X(x)}{\|X(x)\|}\,.$$

Since $\sphericalangle(E^s_x, X(x)) \geq \sphericalangle(E^s_x, E^{cu}_x)$ and the latter is uniformly bounded from zero, we have $|a_x| \geq \kappa$ for some $\kappa > 0$ which depends only on the flow. Then

$$\begin{aligned}\|DR(x)\, e^s_x\| &= \|P_{R(x)} \circ \big(DX^{t(x)}(x) \cdot e^s_x\big)\| \\ &= \frac{1}{|a_x|}\left\|P_{R(x)} \circ \Big(DX^{t(x)}(x)\big(e^s_x - b_x\frac{X(x)}{\|X(x)\|}\big)\Big)\right\| \\ &= \frac{1}{|a_x|}\left\|P_{R(x)} \circ \big(DX^{t(x)}(x) \cdot \hat{e}^s_x\big)\right\| \leq \frac{K}{\kappa}\lambda^{t(x)} \leq \frac{K}{\kappa}\lambda^{t_1}.\end{aligned} \tag{7}$$

Once more it suffices to take t_1 large to ensure that the right hand side is less than λ. □

Given a cross-section Σ, a positive number ρ, and a point $x \in \Sigma$, we define the unstable cone of width ρ at x by

$$C^u_\rho(x) = \{v = v^s + v^u : v^s \in E^s_\Sigma(x),\ v^u \in E^{cu}_\Sigma(x) \text{ and } \|v^s\| \leq \rho\|v^u\|\} \tag{8}$$

(we omit the dependence on the cross-section in our notations).

Let $\rho > 0$ be any small constant. In the following consequence of Proposition 8.1 we assume the neighborhood U_0 has been chose sufficiently small, depending on ρ and on a bound on the angles between the flow and the cross-sections.

Corollary 8.3. *For any $R : \Sigma \to \Sigma'$ as in Proposition 8.1, with $t(\cdot) > t_1$, and any $x \in \Sigma$, we have $DR(x)(C^u_\rho(x)) \subset C^u_{\rho/2}(R(x))$ and*

$$\|DR_x(v)\| \geq \frac{5}{6}\lambda^{-1} \cdot \|v\| \quad \textit{for all} \quad v \in C^u_\rho(x).$$

Proof. Proposition 8.1 immediately implies that $DR_x(C^u_\rho(x))$ is contained in the cone of width $\rho/4$ around $DR(x)\big(E^{cu}_\Sigma(x)\big)$ relative to the splitting

$$T_{R(x)}\Sigma' = E^s_{\Sigma'}(R(x)) \oplus DR(x)\big(E^{cu}_\Sigma(x)\big).$$

(We recall that E^s_Σ is always mapped to $E^s_{\Sigma'}$.) The same is true for E^{cu}_Σ and $E^{cu}_{\Sigma'}$, restricted to Λ. So the previous observation already gives the conclusion of the first part of the corollary in the special case of points in the attractor. Moreover to prove the general case we only have to show that $DR(x)\big(E^{cu}_\Sigma(x)\big)$ belongs to a cone of width less than $\rho/4$ around $E^{cu}_{\Sigma'}(R(x))$. This is easily done with the aid of the flow invariant cone field C^{cu}_a as follows. On the one hand,

$$DX^{t(x)}\big(E^{cu}_\Sigma(x)\big) \subset DX^{t(x)}\big(E^{cu}_x\big) \subset DX^{t(x)}\big(C^{cu}_a(x)\big) \subset C^{cu}_a(R(x))\,.$$

We note that $DR(x)\big(E^{cu}_\Sigma(x)\big) = P_{R(x)} \circ DX^{t(x)}\big(E^{cu}_\Sigma(x)\big)$. Since $P_{R(x)}$ maps $E^{cu}_{R(x)}$ to $E^{cu}_{\Sigma'}(R(x))$ and the norms of both $P_{R(x)}$ and its inverse are bounded by some constant K (see Remark 8.2), we conclude that $DR(x)\big(E^{cu}_\Sigma(x)\big)$ is contained in a cone of width b around $E^{cu}_{\Sigma'}(R(x))$, where $b = b(a, K)$ can be made arbitrarily small by reducing a. We keep K bounded, by assuming the angles between the cross-sections and the flow are bounded from zero and then, reducing U_0 if necessary, we can make a small so that $b < \rho/4$. This concludes the proof since the expansion estimate is a trivial consequence of Proposition 8.1. □

As usual a *curve* is the image of a compact interval $[a, b]$ by a C^1 map. We use $\ell(\gamma)$ to denote its length. By a *cu-curve* in Σ we mean a curve contained in the cross-section Σ and whose tangent direction is contained in the unstable cone $T_z\gamma \subset C^u_\rho(z)$ for all $z \in \gamma$. The next lemma says that *the length of cu-curves linking the stable leaves of nearby points x, y must be bounded by the distance between x and y.*

Lemma 8.4. *Let us we assume that ρ has been fixed, sufficiently small. Then there exists a constant κ such that, for any pair of points $x, y \in \Sigma$, and any cu-curve γ joining x to some point of $W^s(y, \Sigma)$, we have $\ell(\gamma) \leq \kappa \cdot d(x, y)$.*

Here d is the intrinsic distance in the C^2 surface Σ, that is, the length of the shortest smooth curve inside Σ connecting two given points in Σ.

Proof. We consider coordinates on Σ for which x corresponds to the origin, $E^{cu}_\Sigma(x)$ corresponds to the vertical axis, and $E^s_\Sigma(x)$ corresponds to the horizontal axis; through these coordinates we identify Σ with a subset of its tangent space at x, endowed with the Euclidean metric. In general this identification is not an isometry, but the distortion is uniformly bounded, which is taken care of by the constants C_1 to C_4 in what follows.

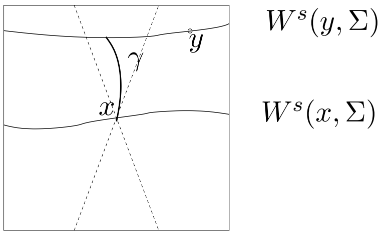

FIGURE 29. The stable manifolds on the cross-section and the *cu*-curve γ connecting them.

The hypothesis that γ is a cu-curve implies that its velocity vector $\dot\gamma(s)$ is contained in the cone of width $C_1 \cdot \rho$ centered at $\gamma(s)$ for all values of the parameter s. In the coordinates described above this means that we may write $\gamma(s) = (\xi(s), s)$ for some C^1 function $\xi : [0, s_0] \to [0, +\infty)$ with $\xi(0) = 0$, $\xi(s) > 0$ for all $s > 0$ and $|\dot\xi| \leq C_1\rho$.

On the other hand, stable leaves are close to being horizontal, that is, fixing some stable leaf through $y \in \Sigma$ we may write it as a graph $(u, \eta(u))$ for a C^1 function $\eta : (-u_0, u_0) \to \mathbb{R}$ with $\eta(0) = d > 0$ and $|\dot\eta| \leq C_2\rho$ (see Figure 29).

Observe now that $h = \eta \circ \xi$ satisfies $|h'| \leq \delta = C_1C_2\rho^2$ and $h(0) = d$, thus $|h(s) - d| \leq \delta \cdot s$ and hence $h(s_*) = 0$ for some $0 < s_* < d/(1-\delta) < d(1+\delta)$.

But this means that

$$\begin{cases} u = \xi(s)) \\ s = \eta(\xi(s)) = \eta(u) \end{cases} \quad \text{or} \quad \gamma(s) = (\xi(s), s) = (u, \eta(u)),$$

that is, we have an intersection between γ and the stable leaf at a distance from x along γ bounded by $d(1+\delta)\sqrt{1+(C_1\rho)^2} < d(1+C_3\rho)$, where C_3 is a constant depending on C_1, C_2 only.

Finally y has coordinates $(\eta(u_1), u_1)$ for some $|u_1| < u_0$ and since $u_0 < \rho$ we get that $\eta(u_1) \geq d - \delta u_1 > d - \delta\rho$ so in Euclidean coordinates $\|x-y\| > d - \delta\rho = d(1-\delta\rho/d)$ and hence $d(x,y) > C_4 d$ for some $C_4 > 0$ depending on all the previous constants (remember that $d < \rho$ also) including the distortion due to the change of metric.

It follows that the length of γ is bounded by $\kappa \cdot d(x,y)$ where $\kappa = (1+\delta)\sqrt{1+(C_1\rho)^2}/C_4$. $\square$

In what follows we take t_1 in Proposition big enough for $\lambda = 1/3$. Note that we will need to decrease λ once taking a bigger t_1.

8.2. **Adapted cross-sections.** Now we exhibit stable manifolds for Poincaré transformations $R : \Sigma \to \Sigma'$. The natural candidates are the intersections $W^s(x,\Sigma) = W^s_\varepsilon(x) \cap \Sigma$ we introduced previously. These intersections are tangent to the corresponding sub-bundle E^s_Σ and so, they are contracted by the transformation. For our purposes it is also important that the stable foliation be invariant:

$$R(W^s(x,\Sigma)) \subset W^s(R(x),\Sigma') \qquad \text{for every } x \in \Lambda \cap \Sigma. \tag{9}$$

In order to have this we restrict our class of cross-sections whose center-unstable boundary is disjoint from Λ. Recall that we are considering cross-sections Σ that are diffeomorphic to the square $[0,1] \times [0,1]$, with the horizontal lines $[0,1] \times \{\eta\}$ being mapped to stable sets $W^s(y,\Sigma)$. The *stable boundary* $\partial^s\Sigma$ is the image of $[0,1] \times \{0,1\}$. The *center-unstable boundary* $\partial^{cu}\Sigma$ is the image of $\{0,1\} \times [0,1]$. The cross-section is *δ-adapted* if

$$d(\Lambda \cap \Sigma, \partial^{cu}\Sigma) > \delta,$$

where d is the intrinsic distance in Σ, see Figure 30. We call *horizontal strip* of Σ the image $h([0,1] \times I)$ for any compact subinterval I, where $h : [0,1] \times [0,1] \to \Sigma$ is the coordinate system on Σ. Notice that every horizontal strip is a δ-adapted cross-section.

In order to prove that adapted cross-sections do exist, we need the following result.

Lemma 8.5. *Let Λ be either a transitive singular-hyperbolic Lyapunov stable set, or a connected singular-hyperbolic attracting set admitting a dense subset of periodic orbits. Then every point $x \in \Lambda$ is in the closure of $W^{ss}(x) \setminus \Lambda$.*

Note that a singular-hyperbolic attractor satisfies the first condition of the statement of Lemma 8.5.

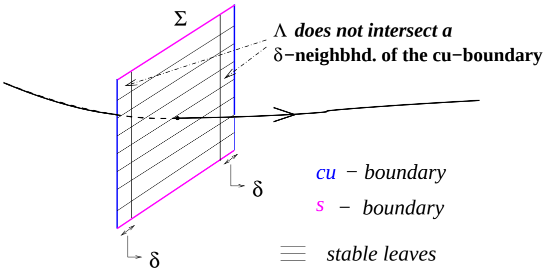

FIGURE 30. An adapted cross-section for Λ.

Proof. The proof is by contradiction. Let us suppose that there exists $x \in \Lambda$ such that x is in the interior of $W^{ss}(x) \cap \Lambda$. Let $\alpha(x) \subset \Lambda$ be its α-limit set. Then

$$W^{ss}(z) \subset \Lambda \quad \text{for every } z \in \alpha(x), \tag{10}$$

since any compact part of the strong-stable manifold of z is accumulated by backward iterates of any small neighborhood of x inside $W^{ss}(x)$. It follows that $\alpha(x)$ does not contain any singularity: indeed, it is known that the strong-stable manifold of each singularity meets Λ only at the singularity (observe that every singularity of Λ is accumulated by regular orbits in Λ).

Therefore the invariant set $\alpha(x) \subset \Lambda$ is hyperbolic. It also follows from (10) that the set

$$H = \overline{\cup\{W^{ss}(y) : y \in \alpha(x) \cap \Lambda\}}$$

is contained in Λ. Also by the same argument as before, this set contains the strong-stable manifolds of all its points. Hence H does not contain any singularity, that is H is uniformly hyperbolic.

We claim that $\overline{W^u(H)}$, the closure of the union of the unstable manifolds of the points of H, is an open set (it is clearly a closed set).

First we show that $W^u(H)$ is open. Note that H contains the whole stable manifold $W^s(z)$ of every $z \in H$: this is because H is invariant and contains the strong-stable manifold of z. Note that the union of the strong-unstable manifolds through the points of $W^s(z)$ contains a neighborhood of z. This proves that $W^u(H)$ is a neighborhood of H. Thus the backward orbit of any point in $W^u(H)$ must enter the interior of $W^u(H)$. Since the interior is, clearly, an invariant set, this proves that $W^u(H)$ is open, as claimed.

Now observe that because $W^u(H)$ is open and invariant, the strong-stable manifold of any $z \in W^u(H)$ is contained in $W^u(H)$, which is contained in Λ since we are assuming that Λ is either Lyapunov stable or attracting. Therefore taking limits we see that $W^{ss}(w) \subset W^u(H)$ for all $w \in \overline{W^u(H)}$. This implies that $\overline{W^u(H)}$ does not contain singularities and is hyperbolic. Finally the unstable manifolds of points in $\overline{W^u(H)}$ are well defined by hyperbolicity and are contained in $\overline{W^u(H)}$, just by taking limits of points in $W^u(H)$. Hence $\overline{W^u(H)}$ contains its stable and unstable manifolds, so it is an open set inside Λ.

Since Λ is also a connected set (which is always the case if Λ is transitive) we obtain $\Lambda = \overline{W^u(H)}$. This means that any singularity $\sigma \in \Lambda$ must be in $\overline{W^u(H)}$, a contradiction. The proof of the lemma is complete. $\square$

Corollary 8.6. *For any $x \in \Lambda$ there exist points $x^+ \notin \Lambda$ and $x^- \notin \Lambda$ in distinct connected components of $W^{ss}(x) \setminus \{x\}$.*

Proof. Otherwise there would exist a whole segment of the strong-stable manifold entirely contained in Λ. Considering any point in the interior of this segment, we would get a contradiction to Lemma 8.5. $\square$

Lemma 8.7. *Let $x \in \Lambda$ be a regular point, that is, such that $X(x) \neq 0$. Then there exists $\delta > 0$ for which there exists a δ-adapted cross-section Σ at x.*

Proof. Fix $\varepsilon > 0$ as in the stable manifold theorem. Any cross-section Σ_0 at x sufficiently small with respect to $\varepsilon > 0$ is foliated by the intersections $W^s_\varepsilon(x) \cap \Sigma_0$. By Corollary 8.6, we may find points $x^+ \notin \Lambda$ and $x^- \notin \Lambda$ in each of the connected components of $W^s_\varepsilon(x) \cap \Sigma_0$. Since Λ is closed, there are neighborhoods $V^\pm$ of $x^\pm$ disjoint from Λ. Let $\gamma \subset \Sigma_0$ be some small curve through x, transverse to $W^s_\varepsilon(x) \cap \Sigma_0$. Then we may find a continuous family of segments inside $W^s_\varepsilon(y) \cap \Sigma_0$, for $y \in \gamma$ with endpoints contained in $V^\pm$. The union Σ of these segments is a δ-adapted cross-section, for some $\delta > 0$, see Figure 31. $\square$

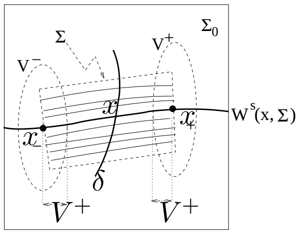

FIGURE 31. The construction of a δ-adapted cross-section for a regular $x \in \Lambda$.

We are going to show that if the cross-sections are adapted, then we have the invariance property (9). Given $\Sigma, \Sigma' \in \Xi$ we set $\Sigma(\Sigma') = \{x \in \Sigma : R(x) \in \Sigma'\}$ the domain of the return map from Σ to Σ'.

Lemma 8.8. *Given $\delta > 0$ and δ-adapted cross-sections Σ and Σ', there exists $T_2 = T_2(\Sigma, \Sigma') > 0$ such that if $R : \Sigma(\Sigma') \to \Sigma'$ defined by $R(z) = R_{t(z)}(z)$ is a Poincaré map with time $t(\cdot) > T_2$, then*

1. *$R\big(W^s(x,\Sigma)\big) \subset W^s(R(x), \Sigma')$ for every $x \in \Sigma(\Sigma')$, and also*
2. *$d(R(y), R(z)) \leq \frac{1}{2} d(y,z)$ for every y, $z \in W^s(x,\Sigma)$ and $x \in \Sigma(\Sigma')$.*

Proof. This is a simple consequence of the relation (7) from the proof of Proposition 8.1: the tangent direction to each $W^s(x,\Sigma)$ is contracted at an exponential rate $\|DR(x)\,e^s_x\| \le Ce^{-\lambda t(x)}$. Choosing T_2 sufficiently large we ensure that

$$Ce^{-\lambda T_2} \cdot \sup\{\ell(W^s(x,\Sigma)) : x \in \Sigma\} < \delta.$$

In view of the definition of δ-adapted cross-section this gives part (1) of the lemma. Part (2) is entirely analogous: it suffices that $Ce^{-\lambda T_2} < 1/2$. □

Remark 8.9. Clearly we may choose $T_2 > T_1$. Remark 8.2 applies to T_2 as well.

The following is a technical consequence of the uniform contraction and the way cross-sections where chosen near real stable leaves.

Lemma 8.10. *Let Σ be a δ-adapted cross-section. Then given any $r > 0$ there exists ρ such that for all $s > 0$, every $y,\, z \in W^s(x,\Sigma)$, and every $x \in \Lambda \cap \Sigma$ we have* $\operatorname{dist}\big(X^s(y), X^s(z)\big) < r$ *if $d(y,z) < \rho$.*

Proof. Let y and z be as in the statement. Then we may find $z' = X^\tau(z)$ in the intersection of the orbit of z with the strong-stable manifold of y satisfying

$$\frac{1}{K} \le \frac{\operatorname{dist}(y,z')}{d(y,z)} \le K \quad \text{and} \quad |\tau| \le K \cdot d(y,z).$$

Then, given any $s > 0$,

$$\begin{aligned}\operatorname{dist}(X^s(y), X^s(z)) &\le \operatorname{dist}(X^s(y), X^s(z')) + \operatorname{dist}(X^s(z'), X^s(z))\\ &\le C \cdot e^{-\lambda s} \cdot \operatorname{dist}(y,z') + \operatorname{dist}(X^{s+\tau}(z), X^s(z))\\ &\le KC \cdot e^{\gamma s} \cdot d(y,z) + K|\tau| \le \big(KC + K^2\big) \cdot d(y,z).\end{aligned}$$

Taking $\rho < r/(KC + K^2)$ we get the statement of the lemma. □

A very useful consequence of the hyperbolicity of Poincaré maps is the following criterion for the existence of a periodic orbit.

Lemma 8.11. *Let $x \in \Lambda$ be a regular point and suppose there exists another regular point $z \in W^{ss}(x) \cap \Lambda$ such that $x \in \omega(z)$. Then x belongs to a periodic orbit.*

Proof. Take an adapted cross-section Σ through x. The conditions on z imply that there exists a Poincaré return map R defined on some substrip $\Sigma(\Sigma)$ containing $W = W^s(x,\Sigma)$, and that this line W is forward invariant $R(W) \subset W$. The contracting property given by Lemma 8.8 ensures there exists a periodic point p for R. Therefore p belongs to a periodic orbit for the flow and to the line W. Hence $z \in W^s(p)$ and so $\omega(z) = \mathcal{O}(p)$, thus $x = p$ since there can be only one intersection $\mathcal{O}(p)$ with Σ on the same stable manifold. □

It is known that any compact invariant subset H of a singular-hyperbolic set Λ is uniformly hyperbolic, and of saddle-type. Using adapter cross sections we can say a bit more.

Lemma 8.12. *Let Λ be a singular-hyperbolic set. Suppose that one of the following conditions is true:*

1. *Λ is Lyapunov stable and transitive;*
2. *Λ is an attractor and H is a compact proper invariant subset of Λ;*
3. *Λ is a attracting set with a dense subset of periodic orbits, H is the set of accumulation points of a branch of the unstable manifold of some singularity σ of Λ, and H does not contain σ.*

Then either $H \subset S(X)$ or, for any adapted cross section Σ through some regular point of H, the intersection $H \cap \Sigma$ is totally disconnected.

Note that the compact invariant set H is covered by a finite number of tubular flow boxes or flow boxes near singularities $U_{\Sigma_i} = X^{(-\varepsilon,\varepsilon)}(\Sigma_i)$, for $\varepsilon > 0$ small and $i = 1, \dots, k$. From Lemma 8.12 we conclude that each $U_{\Sigma_i} \cap H$ has topological dimension one. Hence H in each case of the statement above is a one-dimensional set.

8.3. **Poincaré times near singularities.** Recall that since singularities are Lorenz-like, we have that the unstable manifold $W^u(\sigma_k)$ is one-dimensional, and there is a one-dimensional strong-stable manifold $W^{ss}(\sigma_k)$ contained in the two-dimensional stable manifold $W^s(\sigma_k)$. Most important for what follows, the attractor intersects the strong-stable manifold at the singularity only. Hence for some $\delta > 0$ we may take δ-adapted cross-sections contained $\Sigma^{o,\pm}$ and $\Sigma^{i,\pm}$ in U_0.. Reducing the cross-sections if necessary, i.e. taking $\delta > 0$ small enough, we ensure that the Poincaré times are larger than T_2, so that the same conclusions as in the previous subsections apply here. Indeed using linearizing coordinates it is easy to see that for points $z = (x_1, x_2, \pm 1) \in \Sigma^{i,\pm}$ the time $\tau^\pm$ it takes the flow starting at z to reach one of $\Sigma^{o,\pm}$ depends on x_1 only and is given by

$$\tau^\pm(x_1) = -\frac{\log x_1}{\lambda_1}.$$

We then fix these cross-sections once and for all and define for small $\varepsilon > 0$ the *flow-box*

$$U_{\sigma_k} = \bigcup_{x \in \Sigma^{i,\pm} \setminus \ell^\pm} X_{(-\varepsilon, \tau^\pm(x)+\varepsilon)}(x) \cup (-\delta, \delta) \times (-\delta, \delta) \times (-1, 1)$$

which is an open neighborhood of σ_k with σ_k the unique zero of $X \mid U_{\sigma_k}$. We note that the function $\tau^\pm : \Sigma^{i,\pm} \to \mathbb{R}$ is integrable with respect to the Lebesgue (area) measure over $\Sigma^{i,\pm}$: we say that *the exit time function in a flow box near each singularity is Lebesgue integrable.*

More precisely, we can determine the expression of the Poincaré maps between ingoing and outgoing cross-sections easily thought linearised coordinates

$$\Sigma^{i,+} \cap \{x_1 > 0\} \to \Sigma^{0,+}, \quad (x_1, x_2, 1) \mapsto \big(1, x_2 \cdot x_1^{-\lambda_3/\lambda_1}, x_1^{-\lambda_2/\lambda_1}\big). \tag{11}$$

This shows that the map obtained identifying points with the same x_2 coordinate, i.e. points in the same stable leaf, is simply $x_1 \mapsto x_1^\beta$ where $\beta = -\lambda_2/\lambda_1 \in (0,1)$.

9. A DICHOTOMY FOR 3-DIMENSIONAL FLOWS

In this section we show that a generic C^1 vector field on a closed 3-manifold either has infinitely many sinks or sources or else is singular Axiom A without cycles. These results are

Theorem 9.1. *A generic vector field $X \in \mathfrak{X}^1(M)$ satisfies (only) one of the following properties:*

1. *X has infinitely many sinks or sources.*
2. *X is singular Axiom A without cycles.*

Singular Axiom A means that the non-wandering set of the vector field has a decomposition into finitely many compact invariant sets $\Omega(X) = \Omega_1 \cup \cdots \cup \Omega_k$, each one being either a (uniformly) hyperbolic basic set (i.e. transitive, isolated and with a dense subset of periodic orbits) or a singular-hyperbolic attractor, or a singular-hyperbolic repeller with dense subset of periodic orbits, note that in this decomposition the singular-hyperbolic sets are transitive by definition).

An analogous result was proved by Mañé in [18]

It is known that a generic *non-singular* vector field $X \in \mathfrak{X}^1(M)$ either has infinitely many sinks or sources, or else is Axiom A without cycles, see Mañé [18] or Liao [14]. The robustness of the geometric Lorenz attractor obtained in Section 6 shows that this is not true in general if singularities are allowed. Allowing singularities we can improve this as follows. Let $\mathfrak{V}^1(M) \subset \mathfrak{X}^1(M)$ be the set of vector fields that *cannot be C^1 approximated by homoclinic loops*. The Connecting Lemma implies that any singularity of every $X \in \mathfrak{V}^1(M)$ is separated from the non-wandering set. Using the arguments of Wen [34] and Hayashi [11] we conclude that a generic vector field in $\mathfrak{V}^1(M)$ either has infinitely many sinks or sources or else it is Axiom A without cycles.

Recently Arroyo and Hertz [2] proved that every vector field in $\mathfrak{V}^1(M)$ can be either approximated by one that is Axiom A without cycles, or exhibits a homoclinic tangency associated to a periodic orbit.

Let us describe some consequences Theorem 9.1. The first one is related with the abundance of three-dimensional vector fields exhibiting either attractors or repellers. As noted by Mañé in [18], a generic C^1 diffeomorphism in the 2-sphere S^2 does exhibit either sinks or sources. It is then natural to ask whether such a result is valid for C^1 vector fields in the 3-sphere S^3 instead of C^1 diffeomorphisms in S^2. The answer is negative as the following example shows.

Write $S^3 = \mathbb{R}^3 \cup \{\infty\}$ and consider in $\mathbb{R}^3$ an unknotted two-torus T^2. Then the closure in S^3 of each connected component of $S^3 \setminus T^2$ is a solid two-torus. Consider a Lorenz attractor in one of the solid two-torus and a Lorenz repeller in the other. Since a fundamental domain for the Lorenz attractor (respectively repeller) is an unknotted solid two-torus, we can glue the two solid two-torus through the unknotted torus, obtaining a flow in S^3 whose non-wandering set equals the disjoint union of one Lorenz attractor and one Lorenz repeller. Such a flow is singular Axiom A, and it can not be approximated by vector fields with either sinks or sources. However from Theorem 9.1 we deduce

Corollary 9.2. *A generic vector field in* $\mathfrak{X}^1(M)$ *does exhibit either attractors or repellers.*

The second one is related with a conjecture by Palis in [24], asserting the denseness of vector fields exhibiting a finite number of attractors whose basin of attraction forms a full Lebesgue measure subset. Theorem 9.1 gives an approach to this conjecture in the (open) set $\mathfrak{N}^1(M)$ of C^1 vector fields on a closed 3-manifold M which cannot be C^1 approximated by ones exhibiting infinitely many sinks or sources.

Corollary 9.3. *A generic vector field in* $\mathfrak{N}^1(M)$ *exhibits a finite number of attractors whose basins of attraction form an open and dense subset of* M.

This corollary follows from the no-cycle condition by the classical construction of filtrations adapted to the decomposition of the positive limit set of the flow, as the reader can easily see in [27, Chapter 2 & 3].

Using the filtration to isolate the dynamics around each basic piece of the singular Axiom A decomposition, since the critical elements are robustly hyperbolic nearby each basic piece (recall that singular-hyperbolicity is a robust property of the action of the flow on the tangent bundle), we obtain

Corollary 9.4. *A* C^r *singular Axiom A flow without cycles is in* $\mathcal{G}^r(M)$, *the interior of the set of* C^r *vector fields whose critical elements are hyperbolic, for any* $r \geq 1$.

We note that there exists a classification by Hayashi [10] of the C^1 interior of the set of diffeomorphisms whose periodic points are hyperbolic: they are Axiom A without cycles.

The corresponding result for vector fields is false since the Lorenz attractor is not uniformly hyperbolic. Indeed observe that we can easily construct a singular Axiom A vector field without cycles and with a singular basic set equivalent to the Lorenz attractor: just take the geometric Lorenz attractor constructed in Section 6, and embed and extend this flow to S^3 with a repelling singularity at the north pole and a sink at the south pole.

Proof of Theorem 9.1: The argument is based on the following result. Denote by $\mathfrak{H}^r(M)$ the interior of the set of vector fields $X \in \mathfrak{X}^r(M)$ such that every periodic orbit and singularity of X is hyperbolic, for any $r \geq 1$.

Theorem 9.5. *Generic vector fields in* $\mathfrak{H}^1(M)$ *are singular Axiom A without cycles.*

Following the arguments of Mañé in [18], we can obtain Theorem 9.1 from Theorem 9.5. Indeed, let $\mathfrak{S}^1(M) \subset \mathfrak{X}^1(M)$ be the subset of C^1 vector fields such that every singularity of X is hyperbolic. Then $\mathfrak{S}^1(M)$ is open and dense in $\mathfrak{X}^1(M)$ by the local stability of hyperbolic critical elements. For $X \in \mathfrak{S}^1(M)$ define $A(X)$ to be the set of periodic orbits and singularities of X that are sinks or sources.

The set valued function $\mathfrak{X}^1(M) \ni X \mapsto \overline{A(X)} \in \mathcal{P}(M)$ is lower semicontinuous, again by the local stability of hyperbolic critical elements, where $\mathcal{P}(M)$ denotes the family of compact subsets of M endowed with the Hausdorff distance. Well known topological properties imply that the continuity points $\mathcal{O}$ of this map form a residual subset of $\mathfrak{S}^1(M)$.

This ensures that *every* $X \in \mathcal{O}$ *not satisfying the first item of Theorem 9.1 is in* $\mathfrak{H}^1(M)$.

Indeed for $X_0 \in \mathcal{O}$ with finitely many sinks and sources the set $A(X_0)$ is a finite collection of critical elements of X_0. Assume by contradiction that $X_0 \notin \mathfrak{H}^1(M)$. Then we can find a C^1-near vector field Y with a non-hyperbolic critical element ξ. Hence ξ is away from a neighborhood of $A(X_0)$. However $\mathcal{O} \subset \mathfrak{S}^1(M)$ and $\mathfrak{S}^1(M)$ is open, thus we can take $Y \in \mathfrak{S}^1(M)$. This guarantees that ξ is *not a singularity* of Y. Then the return map to a Poincaré section of the periodic orbit ξ has two eigenvalues, one of which has modulus 1. Perturbing Y we can find $Z \in \mathfrak{S}^1(M)$ arbitrarily C^1-close to Y (and to X_0) having either an attracting or repelling periodic orbit close to ξ. This contradicts the continuity of the set map $A(X)$ at X_0.

Now from Theorem 9.5 there exists a residual set $\mathfrak{R} \subset \mathfrak{H}^1(M)$ such that every vector field in $\mathfrak{R}$ is singular Axiom A without cycles. The class

$$\mathfrak{Y} = \big(\mathcal{O} \setminus \overline{\mathfrak{H}^1(M)}\big) \cup \big(\mathcal{O} \cap \mathfrak{R}\big)$$

is residual in $\mathfrak{X}^1(M)$ by construction (recall that $\mathfrak{S}^1(M)$ is open and dense in $\mathfrak{X}^1(M)$). Note that if $X_0 \in \mathfrak{Y}$ does not satisfy the first item of Theorem 9.1, then $X_0 \in \mathcal{O} \cap \mathfrak{R}$, since X_0 cannot belong to $\mathcal{O} \setminus \overline{\mathfrak{H}^1(M)}$ be the previous claim. This means that X_0 satisfies the second item of the statement of Theorem 9.1. □

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